

Averaging out Inhomogeneous Newtonian Cosmologies: III. The Averaged Navier-Stokes-Poisson Equations

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Abstract

The basic concepts and hypotheses of Newtonian Cosmology necessary for a consistent treatment of the averaged cosmological dynamics are formulated and discussed in details. The space-time, space, time and ensemble averages for the cosmological fluid fields are defined and analyzed with a special attention paid to their analytic properties. It is shown that all averaging procedures require an arrangement for a standard measurement device with the same measurement time interval and the same space region determined by the measurement device resolution to be prescribed to each position and each moment of time throughout a cosmological fluid configuration. The formulae for averaging out the partial derivatives of the averaged cosmological fluid fields and the main formula for averaging out the material derivatives have been proved. The full system of the averaged Navier-Stokes-Poisson equations in terms of the fluid kinematic quantities is derived.

1 Introduction

In order to describe the dynamics of inhomogeneous and anisotropic Newtonian universes one needs to establish a set of averaged Navier-Stokes-Poisson equations in kinematic quantities. To approach this problem the basic concepts and hypotheses of Newtonian Cosmology necessary for a consistent treatment of averaged cosmological dynamics are formulated and discussed in details. The space-time, space, time and ensemble averages for the cosmological fluid fields are defined and analyzed with a special attention paid for their analytic properties. It is shown that all averaging procedures require an arrangement for a standard measurement device with the same measurement time interval and the same

space region determined by the measurement device resolution to be prescribed to each position and each moment of time throughout a cosmological fluid configuration. The formulae for averaging out the partial derivatives of the average fluid fields and the main formula for averaging out the material derivatives have been proved. The full set of the averaged Navier-Stokes-Poisson equations in terms of the fluid kinematic quantities is derived.

The structure of the paper is as follows. The Averaging problem in general relativity which has originated in cosmology when it has been realized that a modification of the Einstein equations by smoothing them out in some appropriate sense should be established for a description of the large-scale structure of the Universe is discussed in Chapter 2. It is shown that Newtonian cosmology also has the Averaging problem, since due to the nonlinearity of the Navier-Stokes-Poisson equations one needs to prove that the averaged Navier-Stokes-Poisson equations have a solution for a homogeneous and isotropic Newtonian universe which can be obtained upon averaging out a locally inhomogeneous and/or anisotropic cosmological fluid configuration representing evolving matter structures. Chapter 3 shows that an averaged description of a Newtonian universe means, from the physical point of view, that the dynamics of the Newtonian universe is written directly in terms of observationally relevant measurable quantities. The Newtonian cosmological macroscopic hypothesis which assumes the existence of two essentially distinct types of temporal and spatial scales, namely, the local scales of cosmological fluid fluctuations and cosmological mean scales, is formulated and discussed in details in Chapter 4. A classification of Newtonian cosmologies is given, in particular, on the basis of this hypothesis. Chapter 5 gives the definitions of the space-time averages of cosmological fluid fields and their correlations and discusses how one can construct an average fluid field by arranging a covering of the Newtonian space-time by a standard measurement device at every moment of time and at every position throughout a cosmological fluid configuration. The Reynolds conditions necessary for any kinds of averages are formulated and proved for the space-time averages in Chapter 6. Properties and conditions on the space-time correlation functions are proved in Chapter 7. Chapter 8 is devoted to the definition and analysis of time averages and time correlation functions. It is shown, in particular, that time averages over an infinite time interval does not depend on time. Chapter 9 is devoted to the definition and analysis of space averages and space correlation functions. It is shown that space averages over the whole space are constant with respect to the space position. The formula for the space-time averaging out of the material derivatives is proved in Chapter 10 and the form of the space-time averaged Navier-Stokes operator is derived. In Chapter 11 a possibility to use the Reynolds transport theorem for a definition of space averages is analyzed and it is shown that one cannot take this approach for the proper treatment of space averages. In Chapter 12 the ensemble averages and ensemble correlation functions are defined and the statistical treatment of cosmological fluid fields is discussed. The Hypothesis of the statistical nature of Newtonian universes is also formulated here. Properties and conditions on the ensemble averages and ensemble correlation functions, such as the Reynolds conditions, the formula for the ensemble averaging out of the material derivatives and the form the ensemble averaged Navier-Stokes operator, are

proved in Chapter 13. The Ergodic hypothesis of Newtonian cosmology is discussed and formulated in Chapter 14. The full set of the averaged Navier-Stokes-Poisson equations in terms of the fluid kinematic quantities valid for the space-time and ensemble averages is derived in Chapter 15. It is pointed out that the system must be supplemented by a system of the equations for the single-point second order moments of fluid fields present in the averaged Navier-Stokes-Poisson equations.

The formulae and Sections from the papers I and II [1], [2] will be referred to as (I-XX) and I-X, and (II-XX) and II-X, correspondingly.

Conventions and notations are as follows. All functions $f = f(x^i, t)$ are defined on 3-dimensional Euclidean space E^3 in the Cartesian coordinates $\{x^i\}$ with Latin space indices i, j, k, \dots running from 1 to 3, and t is the time variable. The Levi-Civita symbol ε_{ijk} is defined as $\varepsilon_{123} = +1$ and $\varepsilon^{123} = +1$ and δ^{ij} , δ_j^i and δ_{ij} are the Kronecker symbols. The symmetrization of indices of a tensor $T_{jk}^i = T_{jk}^i(x^l, t)$ is denoted by round brackets, $T_{(jk)}^i = \frac{1}{2}(T_{jk}^i + T_{kj}^i)$, and the antisymmetrization by square brackets, $T_{[jk]}^i = \frac{1}{2}(T_{jk}^i - T_{kj}^i)$. A partial derivative of T_{jk}^i with respect to a spatial coordinate x^i or time t is denoted either by comma, or by the standard calculus notation, $T_{jk,t}^i = \partial T_{jk}^i / \partial t$ and $T_{jk,l}^i = \partial T_{jk}^i / \partial x^l$, the fluid velocity is $u^i = u^i(x^j, t)$, and the material (total) derivative is $\dot{T}_{jk}^i \equiv dT_{jk}^i / dt = \partial T_{jk}^i / \partial t + u^l \partial T_{jk}^i / \partial x^l = T_{jk,t}^i + u^l T_{jk,l}^i$. Average values are shown by either the angular brackets, or by a bar over a tensor, $\langle T_{jk}^i \rangle \equiv \bar{T}_{jk}^i$ with the corresponding subscripts to distinguish between space-time, time, space and ensemble averages. The Newton gravitational constant, the velocity of light and the Newtonian cosmological constant are G , c and Λ , respectively.

2 The Averaging Problem in Newtonian Cosmology

Our Universe on the largest scales is assumed to be homogeneous and isotropic though locally, at the scales of stars, galaxies and even clusters of galaxies, it is inhomogeneous because of the matter condensation into these discrete structures. Observational data definitely shows a high degree of isotropy which is supported specifically by measurements of the temperature of the cosmic background radiation. The large-scale spatial homogeneity of the Universe up to scale of order 100-300 Mpc, however, still remains an issue which cannot be considered as undoubtedly proved on the basis of cosmological observations. In addition to uncertainties in the measured data, difficulties in proving this fundamental hypothesis originate also in the way how this data are treated. In particular, some theoretical models for analysis of observational data are based on an assumption about such a homogeneity [3]. Therefore, the general relativistic Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmological models [4]-[8] which are homogeneous and isotropic have not been proved as yet to be precisely realistic to represent the large-scale structure of our Universe even for the present epoch. The same is valid for the Newtonian cosmological models, see Section II-9.

The observations of the large-scale characteristics of cosmological matter and radiation

do measure their averaged values over space regions and time intervals. From physical point of view, one can think of the large-scale cosmological models as representing an averaged picture of the Universe which is highly inhomogeneous on the smaller scales characteristic of typical matter structures. From the mathematical point of view one can think of a cosmological model which is inhomogeneous on some scales to mimic local gravitating matter condensations and which, at the same time, is homogeneous on a large scale in some averaged sense, that is, after application of a smoothing procedure to the physical fields and their governing equations. Such a cosmological model would pretend to be more adequate to describe the local dynamics of the real locally inhomogeneous Universe and its large-scale homogeneous and/or isotropic dynamics. Then the Newtonian cosmological principle, see Section II-6, is related to the large-scale space-time structure of a locally inhomogeneous universe averaged out over the space regions of a size much larger than a typical length of inhomogeneities, say, the curvature radius of local matter structures, and much less than a typical large-scale cosmological length, say, the curvature radius of the Universe due to the mean matter density, and the time intervals much larger than a typical time of change of local cosmological variables governing the dynamics of the matter structures and much less than a typical time of change of cosmological variables governing the evolution of the Universe as a whole. The Averaging problem in the general relativistic cosmology calls for a development of such a theoretical physically reasonable and mathematically rigorous framework where one can formulate a realistic inhomogeneous cosmological model which is homogeneous and/or isotropic on average. The first attempt to carry out a space-time volume averaging out the Einstein equations had been made by Shirokov and Fisher [9]. The Averaging problem has been first apparently formulated by Ellis [10] in the framework of general relativistic cosmology (see [11] for a comprehensive review and references). The first covariant approach for averaging out the Einstein equations over space-time regions has been developed by Zalaletdinov [12]-[14] in the framework of macroscopic gravity, a classical macroscopic theory of gravity with the space-time volume averaged (macroscopic) Einstein equations together with equations for the gravitational correlation functions governing the dynamics of continuous macroscopic matter distributions. The Einstein equations are considered as the microscopic field equations suitable for description of point-like matter distributions [14], [15]. The gravitational correlations have been shown to be very important for the dynamics of the averaged (macroscopic) gravitational field, in particular, when all correlations are assumed to vanish the macroscopic Einstein equations become the Einstein equations of general relativity.

Thus, according to the Averaging problem a modification of the Einstein equations by smoothing them out in some appropriate sense should be established for a description of the large-scale structure of the Universe and an adequate large-scale macroscopic cosmological model should be found as a solution to these equations. The Einstein equations of general relativity then have to describe the local dynamics of the Universe by a suitable cosmological model. Upon an appropriate averaging of this cosmological model one must obtain the macroscopic cosmological model which is homogeneous and/or isotropic on some large cosmological scale.

The same problem exists for Newtonian cosmology. A distribution of stars, galaxies and clusters of galaxies in the real Universe can be represented by a locally inhomogeneous distribution of the cosmological fluid which is governed either by the Navier-Stokes-Poisson equations (II-4), (II-8), (II-9) and (II-10), or the system of Navier-Stokes-Poisson equations in terms of kinematic quantities (II-4), (II-8), (II-12), (II-51)-(II-56), (II-59)-(II-61), (I-64)-(II-66), (II-69), (II-28) and (II-70). Assuming the Newtonian cosmological principle, see Section II-6, valid for the large-scale structure of a Newtonian universe, one questions now whether or not the suitably averaged Navier-Stokes-Poisson equations or the Navier-Stokes-Poisson equations in terms of kinematic quantities have a solution for an homogeneous and isotropic Newtonian universe such as this large-scale homogeneous and isotropic cosmological fluid configuration can be obtained upon averaging out the locally inhomogeneous and/or anisotropic distribution of a cosmological fluid configuration representing evolving matter structures.

The Newtonian averaging problem *Given a locally inhomogeneous and/or anisotropic distribution of a cosmological fluid configuration representing the evolving cosmological matter structures as a solution to the Navier-Stokes-Poisson equations (II-4), (II-8), (II-9) and (II-10), if there exists a homogeneous and/or isotropic Newtonian universe as a solution to the suitably averaged Navier-Stokes-Poisson equations such as the homogeneous and isotropic cosmological fluid configuration can be obtained upon averaging out this locally inhomogeneous and/or anisotropic cosmological fluid configuration. Then this locally inhomogeneous and/or anisotropic Newtonian universe will satisfy the Newtonian cosmological principle for its large-scale structure.*

The importance of study of this difficult challenging problem is evident for both Newtonian and general relativistic cosmologies. At the very fundamental level, the Averaging problem questions the validity of the Newtonian cosmological principle, see Section II-6, as a selection principle for a realistic locally inhomogeneous and/or anisotropic model of the Universe to be globally homogeneous and isotropic on its largest scales. From the mathematical point of view, in general relativity due to the nonlinearity of the Einstein equations one cannot expect this is to be true for an arbitrary inhomogeneous cosmological model. Taking an average of the Einstein field operator for the metric tensor serving as the gravitational potential in general relativity does not commute with taking an average of the metric tensor as it has been first pointed out by Shirokov and Fisher [9] (see also [10]-[12]). In fluid mechanics the same argument is valid for the Navier-Stokes-Poisson equations as it has been known after Reynolds [16]. Reynolds was the first to realize that the nonlinearity of the Navier-Stokes equation (I-84) actually contains a hidden key inside it to the understanding of fluid motion through definition of the correlation functions of fluid velocity and density for analytical study of the dynamics of evolving fluids by means of a suitable time and/or space averaging procedure. In Newtonian cosmology this important argument has never been formulated explicitly. Indeed, taking an average value of the Navier-Stokes field operator, the left-hand side of Eq. (II-10), does not produce

the material derivative of an average of the fluid velocity,

$$\left\langle \frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} \right\rangle \neq \frac{\partial \langle u^i \rangle}{\partial t} + \langle u^j \rangle \frac{\partial \langle u^i \rangle}{\partial x^j} \quad (1)$$

where $\langle \cdot \rangle$ stands for an averaging operator. Thus, similar to the Averaging problem in general relativity one cannot expect that an arbitrary inhomogeneous Newtonian universe would provide a solution to the Newtonian averaging problem. A thorough analysis is required to understand if there exist some locally inhomogeneous Newtonian universes satisfying the Newtonian cosmological principle on their largest scales.

3 Averaging and Space-time Measurements

For an inhomogeneous and/or anisotropic Newtonian universe, the spatial and temporal dependence of the instant values of the cosmological fluid velocity $u^i = u^i(x^k, t)$ are very complex. Moreover, if the cosmological fluid flow is turbulent the values of the fluid velocity field are known to be different each time even if the initial conditions have been set the same for each fluid configuration. Due to this extreme uncertainty and sharp changes in space and time of the cosmological fluid velocity field $u^i(x^k, t)$, the density field $\rho(x^k, t)$ and the pressure field $p(x^k, t)$, it is necessary to use an averaging procedure to introduce more regular mean values of the fluid characteristic quantities instead of real fast and sharp varying dynamical fluid fields. These averaged fields may then be studied analytically by means of usual methods of mathematical physics.

Another justification for using the averaged quantities for characterization of the fluid motion comes from the measurement theory. Indeed, every real act of measurement of the velocity, the density or the pressure of a fluid particle is carried out by an observer for some interval of time $T = t_2 - t_1$ during which the particle is moving to another position $\{x_2^k\} = \{x_1^k + \Delta x^k\}$. The measurement time interval, and, correspondingly, the location where a fluid particle moves during the measurement, depends on the measurement device and conditions. To reduce a measurement error in the determination of a fluid field below an acceptable or required value, a measurement may take a considerable time and the influence of possible random fluctuations in the measured fluid quantity value is then reduced as necessary. The measured value of the fluid field is an average over a measurement time interval T and over a space region $S \subset E^3$ which is determined by the precision of the position resolution of the measurement device. All fluctuations in the quantity under measurement with the temporal variations much less than the measurement time and with the spatial variations of much smaller scales than the measurement device resolution scale are completely suppressed without any trace of them in the measured average value¹. Therefore, a formulation of Newtonian cosmology in terms of space

¹One should bear in mind, however, that it only concerns the linear fluctuations of fluid characteristics. Average values of quadratic and higher-order combinations of fluctuating fields do not vanish in general to bring the fluid moment, or correlation, functions which are essential for understanding of the dynamics of moving fluids [17], [18], see Section 5.

and time averages of the cosmological fluid velocity, density and pressure means, from the physical point of view, that the dynamics of the Newtonian universe is written directly in terms of observationally relevant variables, and all possible physical effects such as fluid correlations have been introduced into the theoretical framework during derivation of such an averaged theory.

4 Averaging out a Cosmological Fluid Field

A physical picture of an inhomogeneous Newtonian universe is that any fluid particle has different values of its instant velocity $u^i(x^j, t)$, the density $\rho(x^i, t)$ and the pressure $p(x^i, t)$ as it moves moving along its path. All other fluid particles have different values of the instant velocity, the density and the pressure as they move along their paths and as each compared with others. In order to describe now a cosmological fluid configuration in terms of the mean values of fluid characteristics one should be able to construct the average fields for the fluid velocity $\langle u^i(x^k, t) \rangle$, the fluid density $\langle \rho(x^k, t) \rangle$ and the fluid pressure $\langle p(x^k, t) \rangle$. A cosmological fluid configuration with the average characteristics may have a different dynamics as compared with the original fluid configuration as far as the average fields are not equal to the original fluid fields,

$$\langle u^i(x^j, t) \rangle \neq u^i(x^j, t), \quad \langle \rho(x^i, t) \rangle \neq \rho(x^i, t), \quad \langle p(x^i, t) \rangle \neq p(x^i, t). \quad (2)$$

An equation of motion $x^i = x^i(\xi^j, t)$ for a fluid particle moving in an inhomogeneous self-gravitating fluid configuration and an equation of motion $X^i = X^i(\xi^j, t)$ for the same fluid particle moving in an averaged self-gravitating fluid configuration are expected to be different because of different solutions of the initial value problems for the the fluid velocity field $u^i(x^j, t)$, see (I-10) and (II-31),

$$\frac{dx^i}{dt} = u^i(x^j, t), \quad x^i(0) = \xi^i, \quad (3)$$

and for the average fluid velocity field $\langle u^i(x^j, t) \rangle$ (2)

$$\frac{dX^i}{dt} = \langle u^i(x^j, t) \rangle, \quad X^i(0) = \xi^i, \quad (4)$$

even if the fluid particle has been given the same initial values $x^i(0) = \xi^i$ and $X^i(0) = \xi^i$ in both fluid configurations. So the paths of the same fluid particle as it is moving within an inhomogeneous cosmological fluid distribution and within the corresponding averaged cosmological fluid distribution have to be different in general.

The question of the definition of the average values of classical physical fields is a delicate one and it has a long history in the classical theory of fields (for a discussion and references see [17], [19], [20] for fluid mechanics, [21]- [24] for the classical macroscopic electrodynamics, and [11], [14], [25] for general relativity). In practice, as it has been pointed out above one usually deals with a space-time average values as resulting from

observations and measurements. Let us consider a quantity f which characterizes the fluid in the framework of the field description. Generally, for an inhomogeneous and/or anisotropic cosmological fluid configuration evolving in time such a quantity f is a function of the Eulerian variables $f = f(x^i, t)$ with nonvanishing temporal and spatial derivatives

$$\frac{\partial f}{\partial t} \neq 0, \quad \frac{\partial f}{\partial x^i} \neq 0, \quad (5)$$

which define a temporally and spatially varying field $f(x^i, t)$ for every fluid particle. To provide a nontrivial averaged cosmological dynamics the cosmological fluid field $f(x^i, t)$ is assumed to satisfy the following physical hypothesis:

The Newtonian cosmological macroscopic hypothesis *Given a measurement time interval T and a space region $S \subset E^3$ with a volume V_S determined by the measurement device resolution, a cosmological fluid field $f(x^i, t)$ is supposed to have, at least, either two essentially different temporal variation scales (λ_T, L_T) , or two essentially different spatial variation scales (λ_S, L_S) , or both pairs of scales simultaneously, such as*

$$\lambda_T \ll T \ll L_T, \quad \lambda_S \ll V_S^{-1/3} \ll L_S \quad (6)$$

where λ_T and λ_S are the temporal and spatial scales of local fluid field fluctuations, and L_T and L_S are the temporal and spatial mean cosmological scales, correspondingly.

Since there is in fact a hierarchy of distinct physical scales in our Universe [10], the validity of the Newtonian cosmological macroscopic hypothesis for a fluid field $f(x^i, t)$ should be checked for each pair of scales of interest.

Hypotheses analogous to the Newtonian cosmological macroscopic hypothesis (6) are always assumed, very often implicitly, for every time, space, or space-time averaging scheme (see [17], [19] for fluid mechanics, [21]-[24] for the classical macroscopic electrodynamics, and [11], [14], [25] for general relativity). The physical picture of a Newtonian universe satisfying the Newtonian cosmological macroscopic hypothesis (6) corresponds to our intuitive understanding of our Universe based on observations and space experiments. The Universe is seen now as highly inhomogeneous on the scales of different matter structures which are represented by temporal and/or spatial scales of local fluid field fluctuations λ_T AND λ_S . On its largest scales the Universe shows a definite tendency to homogeneity and isotropy which are represented by temporal and/or spatial mean cosmological scales, L_T AND L_S . On some stages of the evolution of our Universe one can expect, however, that the hypothesis (6) may not be valid or has to be modified. For instance, a consideration of the Universe evolution at early times when it might have had a turbulent regime, the cosmological fluid may have more complicated fluctuation frequency profiles. In this situation, using a mean field picture and the corresponding averages should thoroughly investigated to verify whether or not such a consideration is physically relevant and adequate, see Sections II-6 and 2.

In dependence of whether the Newtonian cosmological macroscopic hypothesis (6) valid or not for a cosmological fluid distribution, one can define three possible classes of Newtonian universes.

Class LC (Locally inhomogeneous and/or anisotropic Newtonian universes with the large-scale cosmological dynamics) *Such Newtonian universes satisfy the Newtonian cosmological macroscopic hypothesis (6) and the averaged cosmological fluid fields $\langle f(x^i, t) \rangle$ has a nontrivial dynamics with either the characteristic temporal variation scale λ_T , or the spatial variation scale λ_S , or with both scales of local fluid field fluctuations with the condition (2) being held.*

This class of Newtonian universes, if it is not empty, would provide a solution to the Averaging problem, see Section 2. One of the Newtonian cosmological models of this class may be a realistic cosmological model of our Universe in the framework of Newtonian cosmology.

Class L (Locally inhomogeneous and/or anisotropic Newtonian universes without the large-scale cosmological dynamics) *If a cosmological fluid configuration does not possess any temporal and/or spatial mean cosmological scales, L_T and L_S , when the averages of the cosmological fluid fields vanish and there is no large-scale cosmological regime of the evolution of such Newtonian universes, that is,*

$$\langle u^i(x^j, t) \rangle = 0, \quad \langle \rho(x^i, t) \rangle = 0, \quad \langle p(x^i, t) \rangle = 0. \quad (7)$$

This class of Newtonian universes has clearly no physical interest when one is looking for a realistic cosmological model. However, it is of interest in connection with the Averaging problem since, if it is found, one may be able to formulate a set of definite constraints on the local matter distribution necessary for a Newtonian universe to belong to the Class LC.

Class C (Newtonian universes with the large-scale cosmological dynamics only) *If a cosmological fluid configuration does not possess any temporal and/or spatial scales of local fluid field fluctuations λ_T and λ_S , when no nontrivial averaged regime is possible and the average cosmological fluid fields are equal to the original fluid fields, that is,*

$$\langle u^i(x^j, t) \rangle = u^i(x^j, t), \quad \langle \rho(x^i, t) \rangle = \rho(x^i, t), \quad \langle p(x^i, t) \rangle = p(x^i, t). \quad (8)$$

This class of Newtonian universes does not seem to be realistic since our Universe does have temporal and/or spatial scales of local fluid field fluctuations λ_T AND λ_S , though a locally inhomogeneous matter distribution. Again, it is of interest in connection with the Averaging problem since, if this Class is found, one may be able formulate a set of definite constraints on the large-scale cosmological dynamics necessary for a Newtonian universe to belong to the Class LC.

As a fundamental physical consequence of the Newtonian cosmological macroscopic hypothesis (6), an averaging region $S \times T$ of a space-time measurement is considered effectively as a single “point” for the averaged cosmological fluid field $\langle f(x^i, t) \rangle$. Such

regions have been called “physically infinitesimally small” by Lorentz [21] and the results of the measurements must be insensitive to a choice of a reference point (x^i, t) to which the average $\langle f(x^i, t) \rangle$ is prescribed, $t \in T$, $\{x^k\} \in S$. Under the Newtonian cosmological macroscopic hypothesis (6) any measurements of cosmological fluid quantities over a time interval T and a space region $S \subset E^3$ with a volume V_S determined by the measurement device resolution guarantee that the errors made while performing such measurements in different positions $\{x^i\}$ within the region S , $\{x^k\} \in S$, and different instants of time t within the interval T , $t \in T$, are always less than $\mathcal{O}(\lambda_T/T)$ and $\mathcal{O}(\lambda_S/V_S^{-1/3})$ (see [17] for a discussion and references therein). The possibility of choosing the averaging time interval T and a space region $S \subset E^3$ with a volume V_S to be intermediate between the fluctuating and mean fluid field scales, λ_T , λ_S and L_T , L_S , assumes that the cosmological fluid motion may be resolved into a smoothly varying averaged motion with a very irregular fluctuating motion superimposed on it. There is a considerable gap between the frequency range characteristic for the mean motion and fluctuating motion.

In studying the averaged (mean) cosmological fluid configurations there are three following types of averages of a cosmological fluid field $f(x^i, t)$ of interest: a time average $\langle f(x^i, t) \rangle_T$, a volume space average $\langle f(x^i, t) \rangle_S$ and an ensemble, or statistical, average $\langle f(x^i, t) \rangle_E$. The first two types of averages can be naturally defined and treated together as a space-time average $\langle f(x^i, t) \rangle_{ST}$.

5 The Space-time Averages and Correlations of a Cosmological Fluid

Given a measurement time interval T and a space region $S \subset E^3$ determined by the measurement device resolution, a space-time average of the fluid field $f = f(x^i, t)$ is defined [14], [17], [22], [24], [25] as

The space-time average of a fluid field f

$$\langle f(x^i, t) \rangle_{ST} = \frac{1}{TV_S} \int_T \int_S f(x^i + x'^i, t + t') dt' dV' \quad (9)$$

where V_S is 3-volume of the space region S ,

$$V_S = \int_S dV. \quad (10)$$

The space-time average (9) is a well-defined function ² of a reference moment of the absolute time t , $t \in T$, and of a reference position $\{x^k\}$, $\{x^k\} \in S$, with the space-time

²Sometimes a continuous weighing function vanishing outside the space-time region $S \times T$ is used in the definition (9) where the integration is then carried out over the whole four-dimensional manifold defined for the physical configuration under consideration. Such a weighting function is completely determined by the properties of a measurement device and the experimental conditions.

average $\langle f(x^i, t) \rangle_{ST}$ being prescribed to (x^i, t) . If the Newtonian cosmological macroscopic hypothesis (6) is satisfied, the choice of the reference time and position is arbitrary within the averaging region $T \times S$. The average value is generally a functional of T and V_S .

Because the Navier-Stokes-Poisson equations are nonlinear one faces necessity to deal with averaging products of fluid fields. The space-time average (9) for a product of cosmological fluid fields $f = f(x^i, t)$ and $h = h(x^i, t)$ defines the so-called correlation function.

The two-point second order space-time correlation, or moment, function of fluid fields f and h *The two-point second order correlation function is the space-time average of a product of $f(x^i, t)$ and $h(y^j, s)$*

$$\langle f(x^i, t)h(y^j, s) \rangle_{ST} = \frac{1}{TV_S} \int_T \int_S f(x^i + x'^i, t + t') h(y^j + x'^j, s + t') dt' dV'. \quad (11)$$

The correlation function (11) is a well-defined function of the reference moments t and s of the absolute time, $t, s \in T$, and of the reference positions $\{x^k\}$ and $\{y^k\}$, $\{x^k\}, \{y^k\} \in S$, with the space-time average $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$ being prescribed to the pair of Eulerian variables (x^i, t) and (y^j, s) and symmetric in these variables

$$\langle f(x^i, t)h(y^j, s) \rangle_{ST} = \langle f(y^j, s)h(x^i, t) \rangle_{ST} \quad (12)$$

From the physical point of view the two-point space-time average (11) corresponds to a simultaneous measurement of f and h by an observer with a measurement device with time measurement interval T and a measurement device resolution region $S \subset E^3$. As far as the Newtonian cosmological macroscopic hypothesis (6) is satisfied for the cosmological fluid configuration under consideration a choice of the reference times t and s and of the reference positions $\{x^k\}$ and $\{y^k\}$ is arbitrary within T and S . The average products of fluid fields are known [17], [16], [26], [27] to be fundamental characteristic functions responsible for essentially nonlinear phenomenon in evolving fluids, such as, for example, turbulence and hydrodynamic instability. If some of such functions do not vanish for a fluid configuration, then the corresponding fluid fields are said to have correlations.

The fluid field correlations *A cosmological fluid configuration has a central two-point second order space-time correlation, or moment, function $C_{ST}^{(2)}(x^i, t; y^j, s)$, if there are, at least, two cosmological fluid fields $f(x^i, t)$ and $h(x^i, s)$ such that*

$$C_{ST}^{(2)}(x^i, t; y^j, s) = \langle f(x^i, t)h(y^j, s) \rangle_{ST} - \langle f(x^i, t) \rangle_{ST} \langle h(y^j, s) \rangle_{ST} \neq 0, \quad (13)$$

$$C_{ST}^{(2)}(x^i, t; y^j, s) = C_{ST}^{(2)}(y^j, s; x^i, t) \quad (14)$$

in some open space region $U \subset E^3$ for an interval of time Δt , $x^k, y^k \in U$, $t, s \in \Delta t$.

One can define the central multi-point higher-order space-time correlation functions $C_{ST}^{(3)}(x^i, t; y^j, s; z^k, r)$ and so on, whenever it is necessary for analysis of the dynamics of a Newtonian universe by making definitions similar to (11) and (13).

In order to characterize the space-time averaged fluid configuration one needs to determine now an average field $\langle f(x^i, t) \rangle_{ST}$ and a correlation function field $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$ for each moments of time t and s at all positions $\{x^k\}$ and $\{y^k\}$. From the physical point of view, that means that it is possible, at least in principle, to carry out the measurements of the quantities f and h for all instants of time during the evolution of a fluid configuration and at its each space point. This requires additional assumptions concerning the measurement interval T and the space region S which are usually made only tacitly, or they are supposed to be trivial (see a discussion in [22], [25]). This is, however, a very significant issue for a correct definition of analytical properties for an average field (see [12], [14], [25] for a detailed discussion).

The First condition (A covering of the Newtonian space-time by averaging time intervals and space regions) *A measurement time interval T and a space region S must be prescribed at every moment of time t and every point $\{x^k\}$ in order to define an averaged fluid field.*

Given such a covering by the measurement interval T and the space region S , the set of the average values calculated at each moment of time t and at each position $\{x^k\}$ will form average fields $\langle f(x^i, t) \rangle_{ST}$ and $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$. As noted above, this set corresponds to the average values of f as they are measured for all instants of time during the evolution of a cosmological fluid configuration at its each space point.

The Second condition (Typical averaging time intervals and space regions) *All time intervals T and the space regions S are typical in some defined sense.*

They are usually required to be of the same shape and volume, $T = \text{const}$ and $V_S = \text{const}$, such as

$$\frac{\partial T}{\partial t} = 0, \quad \frac{\partial T}{\partial x^i} = 0, \quad \frac{\partial V_S}{\partial t} = 0, \quad \frac{\partial V_S}{\partial x^i} = 0, \quad (15)$$

at each moment of time t and at each position $\{x^k\}$. For a Newtonian universe which has the Newtonian space-time geometry, see Sections II-3 and II-5, one can always satisfy both conditions by placing a measurement device at all positions $\{x^k\}$ in E^3 at an initial moment of time $t = t_0$ and by the Lie-dragging of the time interval T and the region S along the congruences of the Cartesian coordinate lines x^k and the absolute time t to get a covering of the manifold by the regions S of the same shape and volume and the time intervals of the same length T at every point of the Newtonian space-time. Another way of arranging such a covering is to associate a measurement device to all positions $\{x^k\}$ in E^3 at an initial moment of time $t = t_0$ with fluid particles located at $\{x^k\}$ at $t = t_0$ and transport the device to each other position during the evolution of a cosmological fluid configuration by comoving with the corresponding fluid particles as they evolve. Both prescriptions must bring the same covering, that is, associate a

standard measurement device with the measurement time interval T and the measurement resolution space region S with each point $\{x^k\}$ at each instant of time t throughout the cosmological fluid configuration. Three issues here are of particular importance for notice:

(A) Though the second procedure involving transportation along cosmological fluid paths of moving fluid particles seem to be more physical, one should bear in mind that this way of construction of a field of the measurement devices relies on the dynamical evolution of a cosmological fluid. In many case it does not cause any complications. However, in situations when the dynamics of an inhomogeneous Newtonian universe exhibits some specific fluid path configurations, like asymptotic attractors, there might be some points in the Newtonian space-time where no measurement device can be brought by this procedure. The first procedure is essentially kinematic and relies only on the topological and differential structure of the underlying Newtonian space-time manifold for an inhomogeneous Newtonian universe. The Eulerian variables (x^i, t) are defined globally on the Newtonian space-time manifold, therefore by the Lie-dragging of the measurement devices along the coordinate lines (x^i, t) they can be determined for all times t and positions $\{x^k\}$. The Lie-dragging itself is a physically well-defined and meaningful procedure since it preserves physical scales along the coordinates lines (x^i, t) and does not affect the manifold symmetries and structure. It does fit perfectly for the purpose of the construction of a covering by averaging regions without any possible dynamical restrictions.

(B) Since time in Newtonian cosmology t is absolute, all clocks of any observers in an inhomogeneous Newtonian universe are always synchronized. So arranging the same measurement time intervals T by such a covering is undoubtedly possible. The situation with a space region $S \subset E^3$ with a volume V_S determined by the measurement device resolution is more delicate. A typical argument against a possibility to arrange a covering with space region S of the same shape and volume makes use of the fact that an evolving cosmological fluid configuration may expand or shrink during its evolution. However, one can easily show that this argument does not even appear if all physical hypotheses have been clearly stated and checked. Indeed, in accordance with the Newtonian cosmological macroscopic hypothesis (6) the characteristic scale $V_S^{-1/3}$ of the space regions S must be always much less than a characteristic cosmological scale L_S , $V_S^{-1/3} \ll L_S$. Therefore any peculiarities of the cosmological fluid evolution of a characteristic cosmological scale L_S cannot affect the measurement device resolution and change either the shape or the volume of the averaging space regions S placed at different positions at different times. Therefore, such a covering by the same space regions S is always possible.

(C) The values of the space-time averaged fluid field $\langle f(x^i, t) \rangle_{ST}$ calculated for such a covering depend now on T and V_S as free parameters and such an average field becomes effectively a local single-valued function of the reference time t and position $\{x^k\}$ only, and the correlation function $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$ becomes a local single-valued function of each pair of variables. This fundamental result and its consequences for the analytical properties of the space-time averages (9) are analyzed in the next Section.

6 The Reynolds Conditions for Average Fluid Fields

In choosing some particular averaging schemes, in addition to the conditions for the construction of the averages fluid fields, see Section 5, one must also investigate a possibility to formulate the general requirements which such a scheme should have. From the point of view of Newtonian cosmology, the most important of these general requirements is, of course, that the application of an averaging rule to the Navier-Stokes-Poisson equations will allow one to get sufficiently simple equations for the average cosmological fluid fields. Reynolds has found such a set of conditions while he was using time averages for the derivation of the averaged Navier-Stokes equations [16]. These conditions have been reformulated later for any averaging procedure applicable in fluid mechanics [17], [20], [28], in the classical macroscopic electrodynamics [21]-[24] and in general relativity [12], [13], [25]. Let us consider fluid quantities $f = f(x^i, t)$ and $h = h(x^i, t)$ which characterize an inhomogeneous and/or anisotropic cosmological fluid configuration evolving in time such that both fluid fields have nonvanishing temporal and spatial derivatives (5). The Reynolds conditions are formulated [17], [20], [28] as follows.

The Reynolds Conditions *For any cosmological fluid fields $f(x^i, t)$ and $h(x^i, t)$ the space-time averages $\langle f(x^i, t) \rangle_{ST}$ and $\langle h(x^i, t) \rangle_{ST}$ must satisfy the following conditions:*

(i) *the space-time averaging is a linear operation*

$$\langle af(x^i, t) + bh(x^i, t) \rangle_{ST} = a \langle f(x^i, t) \rangle_{ST} + b \langle h(x^i, t) \rangle_{ST}, \quad \text{if } a, b = \text{const}, \quad (16)$$

(ii) *the space-time averaging commutes with the partial differentiation*

$$\frac{\partial}{\partial t} \langle f(x^i, t) \rangle_{ST} = \langle \frac{\partial}{\partial t} f(x^i, t) \rangle_{ST}, \quad \frac{\partial}{\partial x^i} \langle f(x^i, t) \rangle_{ST} = \langle \frac{\partial}{\partial x^i} f(x^i, t) \rangle_{ST}, \quad (17)$$

(iii) *the idempotency of the space-time averages*

$$\begin{aligned} \langle \langle f(x^i, t) \rangle_{ST} h(y^i, s) \rangle_{ST} &= \langle f(x^i, t) \rangle_{ST} \langle h(y^i, s) \rangle_{ST} \quad \text{or} \\ \langle \langle f(x^i, t) \rangle_{ST} \rangle_{ST} &= \langle f(x^i, t) \rangle_{ST}. \end{aligned} \quad (18)$$

One can show that the space-time average (9) satisfies the Reynolds conditions, see [12], [13], [25] for discussion and proofs of relevant issues.

Theorem 1 (The Reynolds conditions for the space-time averages) *The space-time average (9) is a linear averaging procedure satisfying the Reynolds condition (16). If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then*

(1*) the space-time average field $\langle f(x^i, t) \rangle_{ST}$ is a local single-valued function of the reference time t and the position $\{x^k\}$, which depends on the value of T and V_S as parameters, such as,

$$\left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \frac{\partial}{\partial x^i} \right) \langle f(x^k, t) \rangle_{ST} = 0, \quad \left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \right) \langle f(x^k, t) \rangle_{ST} = 0, \quad (19)$$

(2*) the space-time average field $\langle f(x^i, t) \rangle_{ST}$ satisfies the Reynolds condition (17).

Proof. By its definition the space-time averaging procedure (9) is a linear operation (16). Let us now assume that in accordance with the First and Second conditions of Section 5 a covering (15) has been arranged for every moment of time t at every position $\{x^i\}$. Then one can determine an averaged field $\langle f(x^i, t) \rangle_{ST}$ and define now a spatial partial derivative of the average field along the coordinate axis $x^{(i)}$ as

$$\frac{\partial}{\partial x^i} \langle f(x^k, t) \rangle_{ST} = \lim_{\Delta x^{(i)} \rightarrow 0} \frac{1}{\Delta x^{(i)}} [\langle f(x^k + \Delta x^{(i)}, t) \rangle_{ST} - \langle f(x^k, t) \rangle_{ST}] \quad (20)$$

where the average value $\langle f(x^i + \Delta x^{(i)}, t) \rangle_{ST}$ is taken at the reference point $\{x^k + \Delta x^{(k)}\}$ and at the moment of time t . Using the definition of the space-time average (9), expanding $f(x^i + \Delta x^{(i)}, t)$ into a Taylor series around the $\{x^k\}$ and taking into account the conditions (15) on the averaging time intervals T and space regions S , one easily derives the commutation formula for the averaging and the spatial derivative,

$$\frac{\partial}{\partial x^i} \langle f(x^k, t) \rangle_{ST} = \langle \frac{\partial}{\partial x^i} f(x^k, t) \rangle_{ST}, \quad (21)$$

which does not contain explicitly either T or V_S as a consequence of the covering (15). Defining a time derivative of the averaged field $\langle f(x^i, t) \rangle_{ST}$ in the same way as (20), one can derive the commutation formula for the averaging and the time derivative,

$$\frac{\partial}{\partial t} \langle f(x^k, t) \rangle_{ST} = \langle \frac{\partial}{\partial t} f(x^k, t) \rangle_{ST}, \quad (22)$$

which does not contain explicitly either T or V_S as a consequence of the covering (15). Calculation of the second derivatives of the averaged field $\langle f(x^i, t) \rangle_{ST}$ in accordance with the definition (20) bring the fundamental result (19). The formula means locality of an average field $\langle f(x^i, t) \rangle_{ST}$, that is, it is a local single-valued function of the reference time t and position $\{x^k\}$, which does depend on the value of T and V_S as parameters. One can work with these averages as with usual functions depending on time t and position $\{x^k\}$ by differentiating and integrating them, expanding into series, solving differential and integral equations for them using standard techniques of mathematical physics. The Reynolds condition (17) is given by (21) and (22). **QED**

A particular choice of the measurement (averaging) time interval T and the measurement resolution (averaging) space region S can be fixed at any convenient stage of analysis. All formalism and the form of the averaged equations do not depend on the

choice of values T and V_S . When this choice is made, the average field $\langle f(x^i, t) \rangle_{ST}$ acquires particular values at each point $\{x^k\}$ at each instant of time t , which reflects the physical conditions of the measurements and the cosmological fluid dynamics.

The last Reynolds condition (18) is known to be an approximate condition [17], [14], [25] which holds in Newtonian cosmology due to the Newtonian cosmological macroscopic hypothesis (6).

Corollary 1 (The idempotency of the space-time averages) *The space-time averages (9) for a cosmological fluid configuration satisfying the Newtonian cosmological macroscopic hypothesis (6) are idempotent and the Reynolds condition (18) holds.*

It should be pointed out here that the physical meaning of the idempotency (18) of the space-time averages (9) is that under the Newtonian cosmological macroscopic hypothesis (6) the measurement time interval T and a space region $S \subset E^3$ with a volume V_S determined by the measurement device resolution guarantee that the errors made to hold the conditions (18) are always less than the errors of the time and space measurements, $\mathcal{O}(\lambda_T/T)$ and $\mathcal{O}(\lambda_S/V_S^{-1/3})$ (see [17] and references therein).

A satisfactory formal analysis of the idempotency property (18) of the space-time, time and space averages is still lacking, though it is extensively used for the derivation of the averaged equations and analysis of various fluid distributions. It has been shown, however, that there are certain classes of volume averaging procedures which satisfy all Reynolds conditions (16)-(18), but these procedures and corresponding averaging kernels do not have a clear physical interpretation and they did not find direct applications in fluid mechanics (see [29], [30] for a discussion and references).

7 The Properties and Conditions for the Correlation Functions

One can now establish the analytic properties of the correlation function $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$ (11) and find its properties analogous to the Reynolds conditions (16)-(18).

Corollary 2 (The properties of the space-time correlation functions) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then the two-point second order moment function (11), $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$, of two cosmological fluid fields $f(x^i, t)$ and $h(x^i, t)$ has the following properties:*

(1*) *it is a bilocal single-valued function of the reference times and positions, (x^k, t) and (y^k, s) , which depends on the value of T and V_S as parameters, such as all second antisymmetrized derivatives with respect to all pairs of the variables $(x^k, t; y^k, s)$ vanish,*

(2*) the two-point second order moment function $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$ satisfies the conditions of the partial differentiation,

$$\begin{aligned} \frac{\partial}{\partial x^k} \langle f(x^i, t)h(y^j, s) \rangle_{ST} &= \left\langle \frac{\partial f(x^i, t)}{\partial x^k} h(y^j, s) \right\rangle_{ST}, \\ \frac{\partial}{\partial t} \langle f(x^i, t)h(y^j, s) \rangle_{ST} &= \left\langle \frac{\partial f(x^i, t)}{\partial t} h(y^j, s) \right\rangle_{ST}, \quad \text{etc} \end{aligned} \quad (23)$$

for all variables $(x^k, t; y^k, s)$,

(3*) the two-point second order moment function $\langle f(x^i, t)h(y^j, s) \rangle_{ST}$ is idempotent

$$\langle \langle f(x^i, t)h(y^j, s) \rangle_{ST} g(z^k, r) \rangle_{ST} = \langle f(x^i, t)h(y^j, s) \rangle_{ST} \langle g(z^k, r) \rangle_{ST} \quad \text{or} \quad (24)$$

$$\langle \langle f(x^i, t)h(y^j, s) \rangle_{ST} \rangle_{ST} = \langle f(x^i, t)h(y^j, s) \rangle_{ST}, \quad (25)$$

where $g = g(z^k, r)$ is another cosmological fluid field.

Therefore one can work with the space-time correlation functions as with usual local functions of many variables depending on reference times and positions by differentiating and integrating them, expanding into series, solving differential and integral equations for them using standard techniques of mathematical physics.

It should be noted here that the conditions (23) mean that a partial differentiation of a correlation function produces a correlation function of the same order including the corresponding partial derivative of a fluid field.

There is an important asymptotic property [17] of the central correlation function $C_{ST}^{(2)}(x^i, t; y^j, s)$ (13), which is of particular interest in Newtonian cosmology.

The asymptotic condition for the central two-point second order space-time correlation function *For any two cosmological fluid fields $f(x^i, t)$ and $h(x^i, t)$ which are correlated (13) and satisfy the Newtonian cosmological macroscopic hypothesis (6), their central two-point second order moment function $C_{ST}^{(2)}(x^i, t; y^j, s)$ tends to zero when the distance between the points $\{x^i\}$ and $\{y^i\}$ and the time interval between t and s infinitely grow*

$$C_{ST}^{(2)}(x^i, t; y^j, s) \rightarrow 0 \quad \text{as} \quad [\delta_{ij}(x^i - y^i)(x^j - y^j)]^{1/2} \rightarrow \infty, \quad |s - t| \rightarrow \infty. \quad (26)$$

This property has a very clear physical meaning stating that cosmological fluid fields are almost independent at extremely remote points in space and for remote moments of time. So the central correlation functions (13) for the cosmological fluid fields in the Newtonian universes of Class LC, see Section 4, always satisfy the asymptotic condition (26). It should be pointed out, however, that for the Newtonian universes of Classes L and C (7) and (8) the central correlation functions (13) may require different asymptotic conditions. The asymptotic conditions on the correlation functions (11) should be considered each time separately for each Newtonian universe.

8 The Time Averages of a Cosmological Fluid

Let us consider now the time averages [17], [20], [31] which are a particular case of the space-time average (9). A time average is defined as a limit of successive measurements of a fluid field $f = f(x^i, t)$ over the measurement time interval T until the fluctuations in its average values become acceptably small.

The time average of a fluid field f

$$\langle f(x^i, t) \rangle_T = \frac{1}{T} \int_{t_1}^{t_2} f(x^i, t + t') dt'. \quad (27)$$

Here the time average (27) is a well-defined function of a reference moment of the absolute time t to which the time average $\langle f(x^i, t) \rangle_T$ is prescribed, $t \in T$, and the position $\{x^i\}$ where the measurements of $f(x^i, t)$ are carried out by an observer with a measurement device resolution region S , $\{x^i\} \in S$. The time average (27) satisfies the Reynolds conditions (16)-(18).

Corollary 3 (The Reynolds conditions for the time averages) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then the time average $\langle f(x^i, t) \rangle_T$ is a local single-valued function (19) of the reference time t and the position $\{x^k\}$ depending on the value of T as a parameter and satisfying the Reynolds conditions (16)-(18):*

(1*) *the time averaging is a linear operation*

$$\langle af(x^i, t) + bh(x^i, t) \rangle_T = a \langle f(x^i, t) \rangle_T + b \langle h(x^i, t) \rangle_T, \quad \text{if } a, b = \text{const}, \quad (28)$$

(2*) *the time averaging commutes with the partial differentiation*

$$\frac{\partial}{\partial t} \langle f(x^i, t) \rangle_T = \langle \frac{\partial}{\partial t} f(x^i, t) \rangle_T, \quad \frac{\partial}{\partial x^i} \langle f(x^i, t) \rangle_T = \langle \frac{\partial}{\partial x^i} f(x^i, t) \rangle_T, \quad (29)$$

(3*) *the time average is idempotent*

$$\langle \langle f(x^i, t) \rangle_T h(y^i, s) \rangle_T = \langle f(x^i, t) \rangle_T \langle h(y^i, s) \rangle_T \quad \text{or} \quad \langle \langle f(x^i, t) \rangle_T \rangle_T = \langle f(x^i, t) \rangle_T. \quad (30)$$

Time averages can be applied for the Newtonian universes when a cosmological fluid distribution satisfies the Newtonian cosmological macroscopic hypothesis (6) for the temporal variation scales (λ_T, L_T) and there is either a local fluid field spatial fluctuation scale λ_S , or a characteristic cosmological spatial scale L_S . In the first case the cosmological fluid configuration is completely inhomogeneous in space with a typical spatial variation scale λ_S of local fluid field fluctuations, to have the vanishing space-time average (9) of a fluid field $f(x^i, t)$ over a space region $S \subset E^3$ with a volume V_S such as $\lambda_S \ll V_S^{-1/3}$,

$$\langle f(x^i, t) \rangle_{ST} = 0, \quad \langle f(x^i, t) \rangle_T \neq 0. \quad (31)$$

In the second case the cosmological fluid configuration is completely spatially homogeneous (49) on the cosmological spatial scale L_S . Then the space-time average (9) of a cosmological fluid field $f(x^i, t)$ over a space region $S \subset E^3$ with a volume V_S such as $V_S^{-1/3} \ll L_S$ does not change it (18) to smooth $f(x^i, t)$ effectively only over the time interval T ,

$$\langle f(x^i, t) \rangle_{ST} = \langle f(t) \rangle_T. \quad (32)$$

Let us now consider the condition when the time average for a cosmological fluid fields $f = f(x^i, t)$ does not depend on time. The following important theorem takes place [17].

Theorem 2 (The time average over an infinite time interval) *For any bounded cosmological fluid quantities $f(x^i, t)$ and $h(x^i, t)$ the time average (27) over an infinitely large time interval T does not depend on time*

$$\lim_{T \rightarrow \infty} \langle f(x^i, t) \rangle_T = \langle f(x^i) \rangle_T. \quad (33)$$

The two-point second order time correlation function $\langle f(x^i, t)h(y^j, s) \rangle_T$ defined due to (11) as

$$\langle f(x^i, t)h(y^j, s) \rangle_T = \frac{1}{T} \int_{t_1}^{t_2} f(x^i, t + t')h(y^j, s + t')dt' \quad (34)$$

and evaluated over an infinitely large time interval, depends on the difference $s - t$ only

$$\lim_{T \rightarrow \infty} \langle f(x^i, t)h(y^j, s) \rangle_T = B(x^i, y^j; s - t). \quad (35)$$

Proof. The proof is as follows. Let us consider the same time average over a measurement time interval T at the reference point t , $\langle f(x^i, t) \rangle_T$, and at another reference point s , $\langle f(x^i, s) \rangle_T$, $s, t \in T$, and estimate the difference $\langle f(x^i, s) \rangle_T - \langle f(x^i, t) \rangle_T$. Due to the idempotency (30) of the times averages this difference is approximately zero since under the Newtonian cosmological macroscopic hypothesis (6) the measurement time interval T guarantee that the errors made to hold the conditions (30) are always less than the errors of the time measurements, $\mathcal{O}(\lambda_T/T)$, see Section 6. However, now one would like to find out a condition when the idempotency holds precisely for the time average defined as (27). Assuming $s > t$ calculation of the difference $\langle f(x^i, s) \rangle_T - \langle f(x^i, t) \rangle_T$ gives

$$\langle f(x^i, s) \rangle_T - \langle f(x^i, t) \rangle_T = \frac{1}{T} \left[\int_{s+t_1}^{s+t_2} f(x^i, r)dr - \int_{t+t_1}^{t+t_2} f(x^i, r)dr \right]. \quad (36)$$

For a bounded function of time $f(x^i, t)$ the difference (36) becomes infinitely small as $|T| \rightarrow \infty$,

$$\langle f(x^i, t) \rangle_T = \langle f(x^i) \rangle_T \quad \text{as} \quad T \rightarrow \infty, \quad (37)$$

which proves (33). The proof of (35) is straightforward if one considers the limit of the correlation function $\langle f(x^i, t)h(y^j, s) \rangle_T$ as $T \rightarrow \infty$

$$B(x^i, y^j; s - t) = \lim_{T \rightarrow \infty} \langle f(x^i, t)h(y^j, s) \rangle_T = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_1}^{t_2} f(x^i, t + t')h(y^j, (s - t) + t + t')dt' \quad (38)$$

which shows that $\langle f(x^i, t)h(y^j, s) \rangle_T$ averaged over an infinitely large time interval can only depend on $(s - t)$, but not on s and t individually. **QED**

The condition (33) can be also easily understood if one observes that a time average over the whole interval of time allowed by the evolution of a Newtonian universe must not depend on time. Indeed, an imaginary measurement of a time dependent physical quantity over the whole time domain where the quantity is determined gives a number provided such a time average value is well-defined. The time correlation function (34) can depend in this case only on the time difference between the correlated values of the fluid fields (35) since the average fields themselves does not depend on time (33). The higher-order time correlation functions can be shown all depend only on the time differences, for example, $B(x^i, y^j, z^k; s - t, r - t) = \lim_{T \rightarrow \infty} \langle f(x^i, t)h(y^j, s)g(z^k, r) \rangle_T$.

A fluid configuration satisfying everywhere the following conditions for averaged fluid fields and their time correlation functions over a finite measurement time interval T

$$\langle f(x^i, t) \rangle_T = \langle f(x^i) \rangle_T, \quad \langle f(x^i, t)h(y^j, s) \rangle_T = B(x^i, y^j; s - t), \quad \text{etc} \quad (39)$$

is called stationary.

The physical meaning of stationarity is transparent since all the equations governing such a fluid motion are time-independent.

The Newtonian stationary universes *A cosmological fluid configuration is stationary if the fluid fields satisfy the conditions (39) at all positions $\{x^k\}$ during its evolution.*

It has been shown in Section II-8 that a particular class of the stationary Newtonian universes, namely, a static homogeneous and isotropic Newtonian universe does not exist if the Newtonian cosmological constant vanishes, $\Lambda = 0$.

9 The Space Averages of a Cosmological Fluid

Let us consider now the space averages [17] which are another particular case of the space-time averages (9). A space average is defined as a measurement of a fluid field $f = f(x^i, t)$ over a space region $S \subset E^3$ with a volume V_S determined by the measurement device resolution such as the fluctuations in its average value become acceptably small.

The space average of a fluid field f

$$\langle f(x^i, t) \rangle_S = \frac{1}{V_S} \int_S f(x^i + x'^i, t) dV' \quad (40)$$

Here the space average (40) is a well-defined function of a reference position $\{x^k\}$ to which the space average $\langle f(x^i, t) \rangle_S$ is prescribed, $\{x^k\} \in S$, and the moment of time t when the measurements of $f(x^i, t)$ are carried out by an observer with a measurement time interval T , $t \in T$. The space average (40) satisfies the Reynolds conditions (16)-(18).

Corollary 4 (The Reynolds conditions for the space averages) *If a covering (15) by the averaging time intervals T and the space regions S is determined, then the space average $\langle f(x^i, t) \rangle_S$ is a local single-valued function (19) of the reference time t and position $\{x^k\}$ depending on the value of V_S as a parameter and satisfying the Reynolds conditions (16)-(18):*

(1*) *the space averaging is a linear operation*

$$\langle af(x^i, t) + bh(x^i, t) \rangle_S = a \langle f(x^i, t) \rangle_S + b \langle h(x^i, t) \rangle_S, \quad \text{if } a, b = \text{const}, \quad (41)$$

(2*) *the space averaging commutes with the partial differentiation*

$$\frac{\partial}{\partial t} \langle f(x^i, t) \rangle_S = \langle \frac{\partial}{\partial t} f(x^i, t) \rangle_S, \quad \frac{\partial}{\partial x^i} \langle f(x^i, t) \rangle_S = \langle \frac{\partial}{\partial x^i} f(x^i, t) \rangle_S, \quad (42)$$

(3*) *the space average is idempotent*

$$\langle \langle f(x^i, t) \rangle_S h(y^i, s) \rangle_S = \langle f(x^i, t) \rangle_S \langle h(y^i, s) \rangle_S \quad \text{or} \quad \langle \langle f(x^i, t) \rangle_S \rangle_S = \langle f(x^i, t) \rangle_S. \quad (43)$$

Space averages can be applied for the Newtonian universes when a cosmological fluid distribution satisfies the Newtonian cosmological macroscopic hypothesis (6) for the spatial variation scales (λ_S, L_S) and there is either a local fluid field temporal fluctuation scale λ_T , or a characteristic cosmological temporal scale L_T . In the first case the cosmological fluid configuration is completely inhomogeneous in time with a typical temporal variation scale λ_T of local fluid field fluctuations to have the vanishing space-time average (9) of a fluid field $f(x^i, t)$ with a time measurement interval T such as $\lambda_T \ll T$

$$\langle f(x^i, t) \rangle_{ST} = 0, \quad \langle f(x^i, t) \rangle_S \neq 0. \quad (44)$$

In the second case the cosmological fluid configuration is stationary (39) on the cosmological spatial scale L_T . Then the space-time average (9) of a cosmological fluid field $f(x^i, t)$ over a time measurement interval T such as $T \ll L_S$ does not change it (18) to smooth $f(x^i, t)$ effectively only over a space region S

$$\langle f(x^i, t) \rangle_{ST} = \langle f(x^i) \rangle_S. \quad (45)$$

Let us now consider the condition when the space average for a cosmological fluid field $f = f(x^i, t)$ does not depend on the reference point $\{x^i\}$. The following important theorem takes place [17].

Theorem 3 (The space averages over the whole space) *For any bounded cosmological fluid quantities $f(x^i, t)$ and $h(x^i, t)$ the space average (40) over the whole space $S = E^3$ does not depend on a position $\{x^i\}$*

$$\lim_{V_S \rightarrow \infty} \langle f(x^i, t) \rangle_S = \langle f(t) \rangle_S. \quad (46)$$

The two-point second order space correlation function $\langle f(x^i, t)h(y^j, s) \rangle_S$ defined due to (11) as

$$\langle f(x^i, t)h(y^j, s) \rangle_S = \frac{1}{V_S} \int_S f(x^i + x'^i, t)h(y^j + x'^i, s)dV' \quad (47)$$

and evaluated over the whole space $S = E^3$ depends on the difference $y^j - x^i$ only

$$\lim_{V_S \rightarrow \infty} \langle f(x^i, t)h(y^j, s) \rangle_S = B(y^j - x^i; t, s). \quad (48)$$

Proof. The proof can be easily accomplished in a completely analogous manner as the proof of the similar properties for the time averages (33) and (35). **QED**

The condition (46) can be also easily understood if one observes that a space average over the whole space E^3 allowed by the definition of a Newtonian universe must not depend on spatial coordinates. Indeed, an imaginary measurement of a position dependent physical quantity over the whole space where the quantity is determined gives a number provided such a space average value is well-defined. The space correlation function (47) can depend in this case only on a difference between the correlated values of the fluid fields (48) since the average fields themselves do not depend on a space position (46). The higher-order space correlation functions can be shown all depend only on the time differences, for example, $B(y^i - x^i, z^j - x^j; t, s, r) = \lim_{V_S \rightarrow \infty} \langle f(x^i, t)h(y^j, s)g(z^k, r) \rangle_S$.

A fluid configuration satisfying everywhere the conditions

$$\langle f(x^i, t) \rangle_S = \langle f(t) \rangle_S, \quad \langle f(x^i, t)h(y^j, s) \rangle_S = B(y^j - x^i; t, s), \quad \text{etc} \quad (49)$$

for the average fluid characteristic fields and their space correlation functions over a space measurement device resolution region $S \subset E^3$ is called homogeneous. The physical meaning of homogeneity is transparent since all the equations governing such a fluid motion are independent of fluid positions $\{x^i\}$.

The Newtonian homogeneous universes *A cosmological fluid configuration is homogeneous if the fluid fields satisfy the conditions (49) at all positions $\{x^k\}$ during its evolution.*

It has been shown in Section II-8 that a particular class of homogeneous Newtonian universes, namely, a static homogeneous and isotropic Newtonian universe does not exist if the cosmological constant vanishes, $\Lambda = 0$. The class of homogeneous and isotropic Newtonian cosmologies has been studied in Sections II-6 and II-12.

It is important to compare the formal definition (49) with the formulation of the Newtonian cosmological principle, see Section II-6, which defines the homogeneity condition

for a cosmological fluid configuration in terms of the data measured by a privileged family of observers. The homogeneity conditions (49) set precise constraints on the character of the averaged fluid fields as they are measured by the observers endowed with measurement devices having the same measurement time interval T and the same measurement device resolution determined by a space region S (15).

10 The Space-time Averaging out of Material Derivatives

In derivation of the averaged systems of the Navier-Stokes-Poisson equations (II-4), (II-8), (II-9) and (II-10), or the Navier-Stokes-Poisson equations in terms of kinematic quantities (II-4), (II-8), (II-12), (II-51)-(II-56), (II-59)-(II-61), (I-64)-(II-66), (II-69), (II-28) and (II-70) one needs to consider carefully the problem of taking an average value of the Navier-Stokes field operator, that is, the left-hand side of Eq. (II-10). As it has been pointed out in the Section 2 an averaging out of this operator (1) does not give the material derivative of an averaged fluid velocity. Let us consider now the space-time averages $\langle u^i(x^k, t) \rangle_{ST}$ (9) of the cosmological fluid velocity $u^i(x^k, t)$ and make use of the Reynolds conditions (16)-(18) after taking the space-time average of the Navier-Stokes field operator, that is, the material derivative of the fluid velocity. One arrives at the expression

$$\begin{aligned} & \frac{\partial \langle u^i(x^k, t) \rangle_{ST}}{\partial t} + \left\langle u^j(x^k, t) \frac{\partial u^i(x^k, t)}{\partial x^j} \right\rangle_{ST} \neq \\ & \frac{\partial \langle u^i(x^k, t) \rangle_{ST}}{\partial t} + \langle u^j(x^k, t) \rangle_{ST} \frac{\partial \langle u^i(x^k, t) \rangle_{ST}}{\partial x^j} \end{aligned} \quad (50)$$

which again is not equal to the material derivative of the averaged velocity $\langle u^i(x^k, t) \rangle_{ST}$ because of the presence of the single-point second order space-time correlation function of the fluid velocity and its spatial derivative $\langle u^j(x^k, t) u^i_{,j}(x^k, t) \rangle_{ST}$. Therefore, a formula for performing averaging out the Navier-Stokes field operator is definitely very important as it will affect the structure of the field operator of the averaged Navier-Stokes equation and finally that of the averaged Navier-Stokes-Poisson equations.

In an inhomogeneous Newtonian universe any fluid particle has different values of its instant velocity $u^i(x^j, t)$, the density $\rho(x^i, t)$ and the pressure $p(x^i, t)$ as it moves along its path. A cosmological fluid configuration with the space-time average fields for the velocity $\langle u^i(x^j, t) \rangle_{ST}$, the density $\langle \rho(x^i, t) \rangle_{ST}$ and the pressure $\langle p(x^i, t) \rangle_{ST}$ may have a different dynamics as compared with the original fluid configuration since the average fields are not equal to the original fluid fields (2) in general, see Section 4. Indeed, a fluid particle moving in an inhomogeneous self-gravitating fluid configuration with the velocity $u^i(x^i, t)$ has an equation of motion (I-1),

$$x^i = x^i(\xi^j, t), \quad (51)$$

as a solution to the initial value problem (3). Due to the Newtonian cosmological macroscopic hypothesis (6), an averaging region $S \times T$ is considered effectively as a single

“point” for the averaged cosmological fluid field $\langle f(x^i, t) \rangle_{ST}$, see Section 4. Therefore, one can consider now the same fluid particle as moving in the averaged self-gravitating fluid cosmological configuration with the average velocity $\langle u^i(x^j, t) \rangle_{ST}$ such as (2)

$$\langle u^i(x^j, t) \rangle_{ST} \neq u^i(x^j, t). \quad (52)$$

The equation of motion

$$X^i = X^i(\xi^j, t) \quad (53)$$

of the fluid particle moving in the averaged cosmological fluid configuration is different from the equation of motion (51) for the fluid particle moving in an inhomogeneous cosmological configuration with the velocity $u^i(x^k, t)$ because a solution to the initial value problem (4),

$$\frac{dX^i}{dt} = \langle u^i(x^j, t) \rangle_{ST}, \quad X^i(0) = \xi^i, \quad (54)$$

is different from that to (3) for the same initial values $x^i(0) = \xi^i$ and $X^i(0) = \xi^i$. So the paths of the same fluid particle as it is considered moving within an inhomogeneous cosmological fluid distribution and within the corresponding averaged cosmological fluid distribution have to be different in general.

A fluid quantity f in the framework of the field description is a function of the Eulerian variables, $f = f(x^i, t)$, and it is also a function of the material variables, $f = f(\xi^i, t)$, such as

$$f = f[x^i(\xi^j, t), t] \quad \text{or} \quad f = f[\xi^i(x^j, t), t], \quad (55)$$

where functions

$$\xi^i = \xi^i(x^j, t) \quad (56)$$

define the initial position $\{\xi^i\}$, $\xi^i = x^i(0)$, of the fluid particle which is at any position $\{x^i\}$ at a moment of time t , see Section I-5. The average fluid field $\langle f \rangle = \langle f(x^i, t) \rangle_{ST}$ is now a function of the Eulerian variables (53) and the material variables

$$\xi^i = \xi^i(X^j, t), \quad (57)$$

such as

$$\langle f \rangle = \langle f \rangle [X^i(\xi^j, t), t] \quad \text{or} \quad \langle f \rangle = \langle f \rangle [\xi^i(X^j, t), t], \quad (58)$$

the functions (58) being related by Eqs. (53) and (57) considered as the laws of change of variables. Then the change in the quantity $\langle f \rangle$ in course of the averaged fluid motion can be characterized by the two different time derivatives

$$\frac{\partial \langle f \rangle}{\partial t} \equiv \frac{\partial \langle f \rangle (X^i, t)}{\partial t} \Big|_{X^i=\text{const}} \quad \text{and} \quad \frac{d \langle f \rangle}{dt} \equiv \frac{\partial \langle f \rangle (\xi^i, t)}{\partial t} \Big|_{\xi^i=\text{const}}. \quad (59)$$

The partial derivative $\partial \langle f \rangle / \partial t$ gives the rate of change of $\langle f \rangle$ with respect to a fixed position $\{X^i\}$, while the material, or convective, derivative $d \langle f \rangle / dt$ measures the rate of change of $\langle f \rangle$ with respect to a moving fluid particle.

The material derivative of the position of a fluid particle defined by the second Eq. (59) for $\langle f \rangle = X^i$ is called its average fluid velocity $\langle u^i \rangle$

$$\langle u^i \rangle (\xi^j, t) = \frac{dX^i}{dt} \equiv \frac{\partial X^i(\xi^j, t)}{\partial t} \Big|_{\xi^i=\text{const}}, \quad \langle u^i \rangle (\xi^j, t) = \langle u^i \rangle (X^j, t) = \langle u^i(x^j, t) \rangle_{ST}. \quad (60)$$

The notion of the average fluid velocity $\langle u^i \rangle$ (60) plays a fundamental role in the field formulation of the fluid mechanics of the averaged flows in the same way as the fluid velocity u^i plays a fundamental role in the field formulation of fluid mechanics of the inhomogeneous flows, see Section II-5.

With the definitions of time derivatives (59) one can calculate the material derivative of any averaged quantity $\langle f \rangle$ as

$$\begin{aligned} \frac{d \langle f \rangle}{dt} &= \left[\frac{\partial}{\partial t} \langle f \rangle (\xi^i, t) \right] \Big|_{\xi^i=\text{const}} = \left\{ \frac{\partial}{\partial t} \langle f \rangle [X^i(\xi^j, t), t] \right\} \Big|_{\xi^i=\text{const}} = \\ &= \frac{\partial \langle f \rangle}{\partial t} \Big|_{X^i=\text{const}} + \frac{\partial \langle f \rangle}{\partial X^i} \left[\frac{\partial X^i(\xi^j, t)}{\partial t} \right] \Big|_{\xi^i=\text{const}} \end{aligned} \quad (61)$$

that can be written in the following form by using the average fluid velocity defined by Eq. (60):

$$\frac{d \langle f \rangle}{dt} = \frac{\partial \langle f \rangle}{\partial t} + \langle u^i \rangle \frac{\partial \langle f \rangle}{\partial X^i} \quad \text{or} \quad \frac{d \langle f \rangle}{dt} = \frac{\partial \langle f \rangle}{\partial t} + \langle u^i \rangle \frac{\partial \langle f \rangle}{\partial x^i}. \quad (62)$$

This formula relates the material and spatial derivatives through the average velocity field in the field picture of the average fluid motion and it expresses the rate of change in $\langle f \rangle$ with respect to a moving fluid particle located at a position $\{X^i\}$ at time t . It is completely analogous to the material derivative of the fluid field $f(x^i, t)$ (I-12), see Section I-6,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i}, \quad (63)$$

where it relates the material and spatial derivatives through the velocity field $u^i(x^j, t)$ in the field description of fluid motion, see Section I-5, and it expresses the rate of change in $f(x^j, t)$ with respect to a moving fluid particle located at a position $\{x^i\}$ at time t .

The Eulerian coordinates (X^i, t) of the averaged cosmological fluid configuration correspond to the Eulerian coordinates (x^i, t) of an inhomogeneous cosmological fluid configuration since the averaged value $\langle f(x^i, t) \rangle_{ST}$ has been prescribed to a reference time t and a position $\{x^i\}$ which are the same moment of time t and the same position $\{X^i\} = \{x^i\}$. One must distinguish between the equations of motion of fluid particles moving in an inhomogeneous fluid distribution and in the corresponding average fluid distribution averaging

out particular relations, but one can always use the spatial coordinates $\{x^i\}$ instead of $\{X^i\}$ in the final averaged formulae (62).

One can now prove the following important theorem establishing the rule for space-time averaging of the material derivatives.

Theorem 4 (The space-time average of the material derivative) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then the space-time average (9) of the material derivative of a cosmological fluid field $f(x^i, t)$ is given by the formula*

$$\left\langle \frac{df}{dt} \right\rangle_{ST} = \left\langle \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i} \right\rangle_{ST} = \frac{d\langle f \rangle}{dt} + \left\langle u^i \frac{\partial f}{\partial x^i} \right\rangle - \langle u^i \rangle \frac{\partial \langle f \rangle}{\partial x^i} \quad (64)$$

where

$$\left\langle u^i \frac{\partial f}{\partial x^i} \right\rangle = \left\langle u^i \frac{\partial f}{\partial x^i} \right\rangle (x^j, t) = \left\langle u^i(x^j, t) \frac{\partial f(x^k, t)}{\partial x^i} \right\rangle_{ST} \quad (65)$$

is the single-point second order space-time correlation function of the fluid velocity $u^i(x^j, t)$ and its spatial derivative $\partial f(x^i, t)/\partial x^i$.

Proof. The proof is straightforward by taking the space-time average (9) of the material derivative of a cosmological fluid field $f = f(x^i, t)$ with using the Reynolds conditions (17), the definitions of the average cosmological fluid field $\langle f \rangle$ (58) and the average cosmological fluid velocity $\langle u^i \rangle$, the definition and expression for the material derivative of the average fluid field (60) and (62) and the introduction of the single-point second order space-time correlation function (65). **QED**

There are two equivalent useful forms of the formula (64):

$$\left\langle \frac{df}{dt} \right\rangle_{ST} = \frac{\partial \langle f \rangle}{\partial t} + \left\langle u^i \frac{\partial f}{\partial x^i} \right\rangle, \quad (66)$$

$$\left\langle \frac{df}{dt} \right\rangle_{ST} = \frac{d\langle f \rangle}{dt} + C^{(2)} \left[u^i, \frac{\partial f}{\partial x^i} \right], \quad C^{(2)} \left[u^i, \frac{\partial f}{\partial x^i} \right] = \left\langle u^i \frac{\partial f}{\partial x^i} \right\rangle - \langle u^i \rangle \frac{\partial \langle f \rangle}{\partial x^i}, \quad (67)$$

where $C^{(2)} [u^i, f_{,i}]$ is the central single-point second order moment function of the fluid velocity $u^i(x^j, t)$ and the spatial derivative $\partial f(x^i, t)/\partial x^j$ of the fluid field $f(x^i, t)$.

The formulae (64)-(67) are of fundamental significance in the derivation of the averaged Navier-Stokes-Poisson equations. An immediate result is their application to the space-time averaging out of the field operator of the Navier-Stokes equation (??).

Corollary 5 (The averaged Navier-Stokes field operator) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological*

fluid configuration, then the space-time average (9) of the field operator of the Navier-Stokes equation (II-10) has the following form:

$$\left\langle \frac{du^i}{dt} \right\rangle_{ST} = \left\langle \frac{\partial u^i}{\partial t} + u^k \frac{\partial u^i}{\partial x^k} \right\rangle_{ST} = \frac{d\langle u^i \rangle}{dt} + \left\langle u^k \frac{\partial u^i}{\partial x^k} \right\rangle - \langle u^k \rangle \frac{\partial \langle u^i \rangle}{\partial x^k} \quad (68)$$

where

$$\left\langle u^j \frac{\partial u^i}{\partial x^j} \right\rangle = \left\langle u^j \frac{\partial u^i}{\partial x^j} \right\rangle (x^k, t) = \left\langle u^j(x^k, t) \frac{\partial u^i(x^k, t)}{\partial x^j} \right\rangle_{ST} \quad (69)$$

is the single-point second order moment function of the fluid velocity $u^i(x^j, t)$ and its spatial derivative $\partial u^i(x^k, t)/\partial x^j$.

Thus, the averaged Navier-Stokes field operator has changed its structure under the space-time averaging (9) as it has been suggested by a heuristic argument (50). The averaging rule (68) plays a central role in establishing the form of the averaged Navier-Stokes-Poisson equations.

11 Averaged Material Derivatives and the Reynolds Transport Theorem

The derivation of the formulae (64)-(67) for a space-time averaging of the material derivatives of cosmological fluid fields has required a thorough analysis of the averaged fluid motion, see Section 10. The foundation of these formulae lies in the possibility to arrange the covering (15). Then in accordance with the First and Second conditions, see Section 5, for a covering of Newtonian space-time by the averaging time intervals T and the space regions S typical in some defined sense, one can determine the space-time average field $\langle f(x^i, t) \rangle_{ST}$ which is a local single-valued function of the reference time t and position $\{x^k\}$, which depends on the value of T and V_S as parameters and satisfies the Reynolds conditions (16)-(18), see Sections 5 and 6. The covering (15) has been shown to be consistent with the Newtonian cosmological macroscopic hypothesis (6) and it associates a standard measurement device with the measurement time interval T and the measurement resolution space region S with each point $\{x^k\}$ at each instant of time t . This is the physical foundation of the Reynolds conditions which are always assumed for derivation of the averaged equations and relations for space-time, time, space and ensemble averages in fluid mechanics [17], [20], [28], the Maxwell macroscopic electrodynamics [21]-[24] and general relativity [12], [13], [25].

Let us now consider a covering of Newtonian space-time by the measurement time intervals T and the space regions $S \subset E^3$ determined by the measurement device resolution, which are taken to be typical in different than the covering (15) sense. If one requires a more general covering

$$\frac{\partial T}{\partial t} = 0, \quad \frac{\partial T}{\partial x^i} = 0, \quad \frac{\partial V_S}{\partial t} \neq 0, \quad \frac{\partial V_S}{\partial x^i} \neq 0 \quad (70)$$

at each moment of time t and position $\{x^k\}$ that means $T = \text{const}$ and $V_S \neq \text{const}$. In fact, it is actually impossible to require a covering other than $T = \text{const}$ with the two first conditions (70) since the time is absolute in Newtonian cosmology. A choice of space regions $S \subset E^3$ such as $V_S \neq \text{const}$ with the two last conditions (70) held is possible formally. Such a choice associates a measurement device with its resolution space region S at each point $\{x^k\}$ at each instant of time t with its characteristic scale $V_S^{-1/3}$ changing in dependence of a position in space. In order to have $V_S \neq \text{const}$, the Newtonian cosmological macroscopic hypothesis (6) should be violated by two possible ways, namely, either $\lambda_S \simeq V_S^{-1/3} \ll L_S$, or $\lambda_S \ll V_S^{-1/3} \simeq L_S$. In the first case the change in the space regions S is due to a spatial scale λ_S of the local cosmological fluid field fluctuations. It cannot be taken seriously from the physical point of view since observers would be affected by the peculiarities of the fast and sharply changing local fields and any space averaging cannot be performed. In the second case the change in the space regions S is due to a spatial cosmological scale L_S and the observers can perform space measurements in principle. However, the measurement space resolution scale $V_S^{-1/3}$ is of the same order as a spatial mean cosmological scale that implies, as a matter of fact, that the space averaging does not resolve any cosmological scale at all. Thus, a covering (70) should result in violation of the Newtonian cosmological macroscopic hypothesis (6) and the impossibility to determine a family of observers with physically reasonable measurement devices which are able to perform proper space-time measurements of the cosmological fluid characteristics.

The thorough analysis of the averaged fluid motion, physical foundations underlying the definition of averages and the formalism for establishing their properties, see Sections 5, 6 and 10, has resulted in the derivation of the formulae (64)-(67) for the space-time averaging out of the material derivatives of cosmological fluid fields. It is especially important for Newtonian cosmology where all these issues have a direct relation to the way how our Universe is observed, how the cosmological data are measured and how a theoretical model takes into account the measurement device characteristics and the conditions of experiments. Using a covering different than (15) and/or using the definitions of a space-time, time or space averages different than (9), (27) and (40) would lead to a violation of the Newtonian cosmological macroscopic hypothesis (6) and an arrangement of a physically unreasonable set of measurement devices. That would affect, as a result, the analytic properties of the averages to abandon the Reynolds conditions (16)-(18) and the formulae (64)-(67).

It is therefore important to analyze possible consequences of such different choices in order to make a comparison with the approach presented here for the formulation of the foundations of Newtonian cosmology and the averaged Navier-Stokes-Poisson equations. Let us consider here a possibility to use the Reynolds transport theorem (I-20) for derivation of the formula for a space averaging out of the material derivatives. On the basis of the Reynolds transport theorem is easy to show the following:

Corollary 6 (The modified Reynolds transport theorem) *The rate of change of*

the ratio of the volume integral $F(t)$ (I-19),

$$F(t) = \int_{\Sigma(t)} f(x^i, t) dV, \quad (71)$$

of the cosmological fluid field $f(x^i, t)$ over an arbitrary closed fluid region $\Sigma(t)$ moving with the fluid to the value $V_{\Sigma}(t)$ of the volume of the region (I-24),

$$\{f\}(t) = \frac{F(t)}{V_{\Sigma}(t)} = \frac{1}{V_{\Sigma}(t)} \int_{\Sigma(t)} f dV, \quad (72)$$

is given by

$$\frac{d\{f\}(t)}{dt} = \left\{ \frac{df}{dt} \right\}(t) + \left\{ f \frac{\partial u^k}{\partial x^k} \right\}(t) - \{f\}(t) \left\{ \frac{\partial u^k}{\partial x^k} \right\}(t). \quad (73)$$

Proof. The formula evidently follows from the Reynolds transport theorem (I-20) and a corollary of the theorem (I-25) giving the rate of change of the volume (I-24) of the region $\Sigma(t)$. **QED**

The function of time $\{f\}(t)$ measures the ratio (72) of the value of the quantity $f(x^i, t)$ inside an arbitrary closed fluid region $\Sigma(t)$ moving with the fluid to the value of the region's volume. The modified Reynolds transport theorem (73) determines the change of the ratio $\{f\}(t)$ rather than of the integral $F(t)$ (71) as the Reynolds transport theorem (I-20) does and they are completely equivalent. All results in analysis of the fluid kinematics made on the basis of (I-20) can be achieved by using the modified Reynolds transport theorem (73).

Though the quantity $\{f\}(t)$ (72) looks similar to a space average $\langle f(x^i, t) \rangle_S$ (40) of the cosmological fluid field $f(x^i, t)$, it cannot be taken as a proper definition of a space average of $f(x^i, t)$ due to the following reasons:

(#1) the quantity $\{f\}(t)$, as well as the quantity $F(t)$, are well-defined functions of t only, since they are defined only for arbitrary closed fluid regions $\Sigma(t)$ moving with the fluid by the meaning of the Reynolds transport theorem (I-20);

(#2) the quantity $\{f\}(t)$ cannot serve as a space average, since no arrangements have been made for the construction of a fluid field $\{f\}(t)$ determined at each point $\{x^i\}$ and each moment of time t , and, therefore, for an arrangement of a set of measurement devices throughout the cosmological fluid configuration, see the First and Second conditions of Section 5;

(#3) the regions $\Sigma(t)$ moving with the fluid change their shape and volume, which raises all the problems with (70) discussed above;

(#4) the Reynolds transport theorem (I-20) is known to be an integral statement fully equivalent to the Euler expansion formula (I-15), see [1] and references therein, therefore any conclusions gained on the basis of application of the modified Reynolds transport

theorem (73) are completely equivalent to the corresponding statements for the integrand of (71).

(#5) the Reynolds transport theorem (I-20) and the modified Reynolds theorem (73) do not take into account by their physical meaning and their conditions that the fluid particles of an averaged cosmological fluid configuration have different equations of motion $X^i = X^i(\xi^j, t)$ (53), see Section 10, as compared with the equations of motion $x^i = x^i(\xi^j, t)$ (51) of the fluid particles of the corresponding inhomogeneous fluid configuration and the region $\Sigma(t)$ is supposed to move along the fluid particles' paths determined by (51);

The modified Reynolds theorem (73) cannot be therefore used as a formula for a space averaging out of the material derivatives instead of the formulae (64)-(67).

12 The Ensemble Averages of a Cosmological Fluid

The use of space-time, time or space averages (9), (27) and (40) is very convenient from the practical point of view, but leads sometimes to analytical difficulties as it has been discussed above. It is desirable therefore to find another method of defining the average fluid fields such that it would have well-defined analytic properties and would be universal. A convenient definition of this type is known to be provided by the statistical picture of the fluid fields as random fields (see [17], [18], [20], [28], [32], [33] and reference therein), which has been established by Richardson [34], [35], Taylor [36], Kolmogorov [37], [38] and Kampé de Fériet [39].

The basic feature of the statistical approach to the motion of fluid is the transition from the consideration of a single moving fluid configuration given by the equations of motion of fluid particles $x^i = x^i(\xi^j, t)$ (51) or $\xi^i = \xi^i(x^j, t)$ (56) to the consideration of the statistical ensemble of all similar moving fluid configurations created by some set of fixed initial and boundary conditions. If the values of the fluid velocity field $f(x^i, t)$ for an evolving inhomogeneous fluid configuration are measured many times under the same set of conditions, the measured values $\langle f(x^i, t) \rangle$ for the same moment of time t and the same position $\{x^i\}$ are expected to be different due to local spatial and temporal fluctuations of $f(x^i, t)$. However, the arithmetic mean of all these values at the moment of time t and the position $\{x^i\}$ can be expected to tend to a definite value if a sufficiently large number of measurements has been carried out. This arithmetic average over all possible realizations of the fluid quantity $f(x^j, t)$, that is, all the values $\langle f(x^j, t) \rangle$ as measured, in principle, by an observer, is called an ensemble average of $f(x^j, t)$ at the time t and the position $\{x^i\}$.

The ensemble average, or the probability mean, of a fluid field f

$$\langle f \rangle_E(x^i, t) = \int_{-\infty}^{+\infty} f P_{(x^i, t)}(f) df. \quad (74)$$

Here the probability density function $P_{(x^i, t)}(f)$ defined for all times t and points $\{x^i\}$ of the cosmological fluid configuration determines the probabilities of different values $f(x^i, t)$

among all measured as

$$P_{(x^i,t)}(f)df = \text{Probability}[f < f(x^i, t) < f + df], \quad \int_{-\infty}^{+\infty} P_{(x^i,t)}(f)df = 1. \quad (75)$$

The quantity Probability $[f < f(x^i, t) < f + df]$ in (75) stands for the probability of $f(x^i, t)$ to have values between f and $f + df$. A fluid field f having a definite probability density is called the random variable and the set of all possible probabilities $P_{(x^i,t)}(f', f'') = \text{Probability}[f' < f(x^i, t) < f'']$ corresponding to f is called its probability distribution.

The ensemble average (74) is a well-defined function of a reference moment t of the absolute time, $t \in T$, and a reference position $\{x^k\}$, $\{x^k\} \in S$, where the ensemble average $\langle f \rangle_E(x^i, t)$ is evaluated by a measurement device with the measurement time interval T and the space region $S \subset E^3$ determined by the measurement device resolution. It should be pointed out here that the measured values of f may be time, space, or space-time averages in dependence on the physical conditions of a cosmological fluid configuration under consideration. The nature of measurements must be taken into account when making physical predictions and interpretations on the basis of the statistical treatment of fluid dynamics.

Because the Navier-Stokes-Poisson equations are nonlinear one faces necessity to deal with averaging products of the fluid fields. The ensemble average (74) for a product of cosmological fluid fields $f = f(x^i, t)$ and $h = h(x^i, t)$ defines the so-called ensemble correlation function.

The two-point second order ensemble correlation, or moment, function of fluid fields f and h *The two-point second order correlation function is the ensemble average of a product of $f(x^i, t)$ and $h(y^j, s)$*

$$\langle fh \rangle_E(x^i, t; y^j, s) = \int_{-\infty}^{+\infty} fh P_{(x^i,t)(y^j,s)}(f, h) df dh. \quad (76)$$

Here $P_{(x^i,t)(y^j,s)}(f, h)$ is the two-dimensional joint probability density function defined for all times t and s and points $\{x^i\}$ and $\{y^i\}$ of the cosmological fluid configuration determines the probabilities of different values $f(x^i, t)$ and $h(y^j, s)$ among all measured as

$$P_{(x^i,t)(y^j,s)}(f, h) = \text{Probability}[f < f(x^i, t) < f + df, h < h(y^j, s) < h + dh],$$

$$\int_{-\infty}^{+\infty} P_{(x^i,t)(y^j,s)}(f, h) df dh = 1. \quad (77)$$

The quantity Probability $[f < f(x^i, t) < f + df, h < h(y^j, s) < h + dh]$ in (77) stands for the probability of $f(x^i, t)$ and $h(y^j, s)$ to have simultaneously values between f and $f + df$ and between h and $h + dh$, correspondingly.

The correlation function (76) is a well-defined function of the reference moments t and s of the absolute time, $t, s \in T$, and of the reference positions $\{x^k\}$ and $\{y^k\}$, $\{x^k\}, \{y^k\} \in S$, with the ensemble average $\langle fh \rangle_E(x^i, t; y^j, s)$ being prescribed to the pair of the Eulerian variables (x^i, t) and (y^j, s) and symmetric in these variables,

$$\langle fh \rangle_E(x^i, t; y^j, s)_E = \langle hf \rangle_E(y^j, s; x^i, t). \quad (78)$$

From the physical point of view the two-point ensemble average (76) corresponds to a simultaneous measurement of f and h by an observer with a measurement device with the time measurement interval T and the measurement device resolution region $S \subset E^3$. As far as the Newtonian cosmological macroscopic hypothesis (6) is satisfied for a cosmological fluid configuration under consideration a choice of the reference times t and s and the reference positions $\{x^k\}$ and $\{y^k\}$ is arbitrary within T and S .

The correlation functions of fluid fields are fundamental characteristic functions responsible for essentially nonlinear phenomenon in evolving fluids, such as, for example, turbulence and hydrodynamic instability [16], [17], [26], [27]. If some of such functions do not vanish for a fluid configuration, then the corresponding fluid fields are said to have correlations.

The fluid field correlations *A cosmological fluid configuration has a central two-point second order ensemble correlation, or moment, function $C_E^{(2)}(x^i, t; y^j, s)$, if there are, at least, two cosmological fluid fields $f(x^i, t)$ and $h(x^i, s)$ such that*

$$C_E^{(2)}(x^i, t; y^j, s) = \langle fh \rangle_E(x^i, t; y^j, s) - \langle f \rangle_E(x^i, t) \langle h \rangle_E(y^j, s) \neq 0, \quad (79)$$

$$C_E^{(2)}(x^i, t; y^j, s) = C_E^{(2)}(y^j, s; x^i, t) \quad (80)$$

in some open space region $U \subset E^3$ for an interval of time Δt , $x^k, y^k \in U$, $t, s \in \Delta t$.

One can define the central multi-point higher-order ensemble correlation functions, $C_E^{(3)}(x^i, t; y^j, s; z^k, r)$ and so on, by using higher-dimensional joint probability density functions $P_{(x^i, t)(y^j, s)(z^k, r)}(f, h, g)$ and so on, whenever it is necessary for analysis of the dynamics of a Newtonian universe by making definitions similar to (76) and (79).

In order to characterize the ensemble averaged fluid configuration one needs to determine now the ensemble averaged fields $\langle f \rangle_E(x^i, t)$ and the correlation function fields $\langle fh \rangle_E(x^i, t; y^j, s)$ for each moments of time t and s at all positions $\{x^k\}$ and $\{y^k\}$. From the physical point of view, that means that it is possible, at least in principle, to carry out the measurements of the quantities f and h for all instants of time during the evolution of a fluid configuration and at its each space position. This requires a covering by the averaging time intervals T and the averaging space regions S to be determined throughout the cosmological fluid configuration in accordance with the First and Second conditions, see Section 5. It will be assumed that a covering (15) has been arranged throughout the cosmological fluid configuration and the Newtonian cosmological macroscopic hypothesis (6) is valid for this set of measurement devices.

Thus one can formulate the main hypothesis on the statistical description of Newtonian universes.

The Hypothesis of the statistical nature of Newtonian universes: *The cosmological fluid fields of the components of the fluid velocity vector $u^i(x^j, t)$, the fluid density $\rho(x^{ki}, t)$, the fluid pressure $p(x^i, t)$ and the Newtonian gravitational potential $\phi(x^i, t)$ are assumed to be random fields defined for all times t and positions $\{x^i\}$ of the cosmological fluid configuration with the corresponding probability densities $P_{(x^j, t)}(u^i)$, $P_{(x^i, t)}(\rho)$, $P_{(x^i, t)}(p)$ and $P_{(x^i, t)}(\phi)$. If the cosmological fluid fields have correlations (79), that is, they are statistically interconnected, it is assumed that there exist the joint probability densities for corresponding fluid fields.*

13 The Properties of the Ensemble Averages

The ensemble average (74) satisfies the Reynolds conditions (16)-(18).

Corollary 7 (The Reynolds conditions for the ensemble averages) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then the ensemble average $\langle f \rangle_E(x^i, t)$ is a local single-valued function*

$$\left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial t} - \frac{\partial}{\partial t} \frac{\partial}{\partial x^i} \right) \langle f \rangle_E(x^i, t) = 0, \quad \left(\frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^j} \frac{\partial}{\partial x^i} \right) \langle f \rangle_E(x^i, t) = 0, \quad (81)$$

of the reference time t and position $\{x^k\}$ and it satisfies the Reynolds conditions (16)-(18): (1) the ensemble averaging is a linear operation*

$$\langle af + bh \rangle_E(x^i, t) = a \langle f \rangle_E(x^i, t) + b \langle f \rangle_E(x^i, t), \quad \text{if } a, b = \text{const}, \quad (82)$$

(2) the ensemble averaging commutes with the partial differentiation*

$$\frac{\partial}{\partial t} \langle f \rangle_E(x^i, t) = \left\langle \frac{\partial f}{\partial t} \right\rangle_E(x^i, t), \quad \frac{\partial}{\partial x^i} \langle f \rangle_E(x^i, t) = \left\langle \frac{\partial f}{\partial x^i} \right\rangle_E(x^i, t), \quad (83)$$

(3) the ensemble average is idempotent*

$$\langle \langle f \rangle_E(x^i, t) h \rangle_E(y^i, s) = \langle f \rangle_E(x^i, t) \langle h \rangle_E(y^i, s) \quad \text{or} \quad \langle \langle f \rangle \rangle_E(x^i, t) = \langle f \rangle_E(x^i, t). \quad (84)$$

Proof. The formulae (81)-(84) follow immediately from the definition of the ensemble average (74). **QED**

The analytic properties of the ensemble averages are the same as those of space-time, time and space averages. It is remarkable fact that they all are rigorous properties for the ensemble averages. This is one of the fundamental advantages of using this type of mean fluid fields.

The formula for the ensemble averaging out of the material derivatives can be proved as follows.

Theorem 5 (The ensemble average of the material derivative) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then the ensemble average (74) of the material derivative of a cosmological fluid field $f(x^i, t)$ is given by the formula*

$$\left\langle \frac{df}{dt} \right\rangle_E = \left\langle \frac{\partial f}{\partial t} + u^i \frac{\partial f}{\partial x^i} \right\rangle_E = \frac{d\langle f \rangle_E}{dt} + \left\langle u^i \frac{\partial f}{\partial x^i} \right\rangle_E - \langle u^i \rangle_E \frac{\partial \langle f \rangle_E}{\partial x^i} \quad (85)$$

where $\langle u^i f, i \rangle_E = \langle u^i f, i \rangle_E(x^j, t)$ is the single-point second order correlation function (76) of the fluid velocity $u^i(x^j, t)$ and the spatial derivative $\partial f(x^i, t)/\partial x^i$, and the material derivative of $\langle f \rangle_E(x^i, t)$ is

$$\frac{d\langle f \rangle_E}{dt} = \frac{\partial \langle f \rangle_E}{\partial t} + \langle u^i \rangle_E \frac{\partial \langle f \rangle_E}{\partial x^i}. \quad (86)$$

Proof. The proof is straightforward by taking the ensemble average (74) of the material derivative (63) of a cosmological fluid field $f = f(x^i, t)$ with using the Reynolds conditions (83), the definitions of the ensemble average $\langle f \rangle_E(x^i, t)$ (74) of the cosmological fluid field and the ensemble average $\langle u^i \rangle_E(x^k, t)$ of the cosmological fluid velocity $u^i(x^j, t)$, and the introduction of the single-point second order correlation function $\langle u^i f, i \rangle_E(x^j, t)$ (76).

QED

There are two equivalent useful forms of the formula (85) which are completely analogous to the formulae (66) and (67).

Corollary 8 (The ensemble averaged Navier-Stokes field operator) *If a covering (15) by the averaging time intervals T and space regions S is determined through the cosmological fluid configuration, then the ensemble average (74) of the field operator of the Navier-Stokes equation (II-10) has the following form:*

$$\left\langle \frac{du^i}{dt} \right\rangle_E = \left\langle \frac{\partial u^i}{\partial t} + u^k \frac{\partial u^i}{\partial x^k} \right\rangle_E = \frac{d\langle u^i \rangle_E}{dt} + \left\langle u^k \frac{\partial u^i}{\partial x^k} \right\rangle_E - \langle u^k \rangle_E \frac{\partial \langle u^i \rangle_E}{\partial x^k} \quad (87)$$

where $\langle u^j u^i, j \rangle_E = \langle u^j u^i, j \rangle_E(x^k, t)$ is the single-point second order correlation function (76) of the fluid velocity $u^i(x^j, t)$ and its spatial derivative $\partial u^i(x^k, t)/\partial x^j$ and the material derivative of the ensemble averaged fluid velocity $\langle u^i \rangle_E(x^j, t)$ is given by (86).

One can now establish the analytic properties of the ensemble correlation function $\langle fh \rangle_E(x^i, t; y^j, s)$ (76) and prove the properties analogous to the conditions (23)-(25).

Corollary 9 (The properties of the ensemble correlation functions) *If a covering (15) by the averaging time intervals T and space regions S is determined through the cosmological fluid configuration, then the two-point second order moment function (76), $\langle fh \rangle_E(x^i, t; y^j, s)$, of two cosmological fluid fields $f(x^i, t)$ and $h(x^i, t)$ has the following properties:*

(1*) it is a bilocal single-valued function of the reference times and positions, (x^k, t) and (y^k, s) , which depends on the value of T and V_S as parameters, such as all second antisymmetrized derivatives with respect to all pairs of the variables $(x^k, t; y^k, s)$ vanish;

(2*) the two-point second order moment function $\langle fh \rangle_E(x^i, t; y^j, s)$ satisfies the conditions of the partial differentiation

$$\begin{aligned}\frac{\partial}{\partial x^k} \langle fh \rangle_E(x^i, t; y^j, s) &= \left\langle \frac{\partial f}{\partial x^k} h \right\rangle_E(x^i, t; y^j, s), \\ \frac{\partial}{\partial t} \langle fh \rangle_E(x^i, t; y^j, s) &= \left\langle \frac{\partial f}{\partial t} h \right\rangle_{EE}(x^i, t; y^j, s), \quad \text{etc}\end{aligned}\tag{88}$$

for all variables $(x^k, t; y^k, s)$;

(3*) the two-point second order ensemble moment function $\langle fh \rangle_E(x^i, t; y^j, s)$ is idempotent

$$\langle \langle fh \rangle_E(x^i, t; y^j, s) g \rangle_E(z^k, r) = \langle fh \rangle_E(x^i, t; y^j, s) \langle g \rangle_E(z^k, r) \quad \text{or}\tag{89}$$

$$\langle \langle fh \rangle \rangle_E(x^i, t; y^j, s) = \langle fh \rangle_E(x^i, t; y^j, s)\tag{90}$$

where $g = g(z^k, r)$ is another cosmological fluid field.

Proof. The formulae (88)-(90) follow immediately from the definition of the ensemble correlation function (76). **QED**

Thus, one can work with the ensemble correlation functions as with usual local functions of many variables depending on the reference times and positions by differentiating and integrating them, expanding into series, solving differential and integral equations for them using standard techniques of mathematical physics.

It should be noted here that the conditions (88) mean that a partial derivative of an ensemble correlation function produces an ensemble correlation function of the same order including the corresponding partial derivative of a fluid field.

There is an important asymptotic property [17] of the central ensemble correlation function $C_E^{(2)}(x^i, t; y^j, s)$ (79), which is of particular interest in Newtonian cosmology.

The asymptotic condition for the central two-point second order ensemble correlation function For any two cosmological fluid fields $f(x^i, t)$ and $h(x^i, t)$ which are correlated (79) and satisfy the Newtonian cosmological macroscopic hypothesis (6), their central two-point second order moment function $C_E^{(2)}(x^i, t; y^j, s)$ tends to zero when the distance between the points $\{x^i\}$ and $\{y^i\}$ and the time interval between t and s infinitely grow

$$C_E^{(2)}(x^i, t; y^j, s) \rightarrow 0 \quad \text{as} \quad [\delta_{ij}(x^i - y^i)(x^j - y^j)]^{1/2} \rightarrow \infty, \quad |s - t| \rightarrow \infty.\tag{91}$$

14 The Ergodic Hypothesis of Newtonian Cosmology

The ensemble averages are undoubtedly more convenient for analytic description of moving fluids. Adopting a hypothesis on the existence of the probability distributions for all fluid fields, such the Hypothesis of the statistical nature of Newtonian universes in Newtonian cosmology, see Section 12, one can use the mathematical techniques of the modern probability theory and statistical mechanics. The ensemble averages (74) and (76) are defined uniquely (81) throughout a cosmological fluid configuration and have all properties naturally required, such as the Reynolds conditions (82)-(84) and the conditions (88)-(90). It is important, however, to note that in the framework of this approach, an important new question arises regarding the comparison of the theoretical deductions following from the statistical treatment of the dynamics of fluids with the data of direct space-time, time and space measurements.

According to its definition (74) the ensemble average is a mean value taken over all possible values of the fluid field under study, which are measured by a measurement device with a time measurement interval T and a measurement space region S determined by the device resolution. Thus, to determine the average values by an experiment or, say, by cosmological observations, with an acceptable high accuracy, one should carry out a large number of measurements in many set of repeated similar observations or experiments. In practice, however, no such multiply repeated measurements are available, or even possible to perform. Therefore in most cases one is forced to determine the average values from the data taken in the course of a single experiment where the data obtained either by a space-time, time or space averaging over a time interval T and/or a space region S . From the physical point of view it means that the assumption about the existence of the probability distributions, see the Hypothesis of the statistical nature of Newtonian universes, see Section 12, does not by itself eliminate the problem of the validity of using the space-time (9), time (27) or space averages (40) for description and analysis of moving inhomogeneous fluids. But in this context the problem has acquired a different formulation and interpretation. Indeed, instead of investigating the particular properties of these averaging procedures, one should establish how close the space-time, time or space averaged values of the fluid fields measured in real observations and experiments are to the corresponding ensemble average values. It is known from the statistical mechanics that a replacement of an ensemble average value of a random quantity calculated over all possible states of this quantity by the directly measurable time average of the same quantity may be only possible if the averaging time interval T becomes infinitely large. Under this condition the time average converges to the corresponding ensemble average [17], [18], [20], [28], [32], [33]. In certain very special cases, the validity of this assumption may be proved rigorously. In the most cases of interest this fundamental physical hypothesis of the consistency between theoretical models, their experimental verification and the real physical phenomenon remains unproved and it is usually adopted as an additional hypothesis called the Ergodic hypothesis. It is formulated here for Newtonian cosmology.

The Ergodic hypothesis of Newtonian cosmology: *If a cosmological fluid configuration satisfies the Newtonian cosmological macroscopic hypothesis (6) and the Hypothesis*

of the statistical nature of Newtonian universes, see Section 12 and a covering (15) by the averaging time intervals T and the space regions S is determined throughout the cosmological fluid configuration, then for any cosmological fluid field $f(x^i, t)$ the following conditions are assumed to hold:

(1) its ensemble average $\langle f \rangle_E(x^i, t)$ (74) is equal to its time average $\langle f(x^i, t) \rangle_T$ (27) over an infinitely large time interval T and both averages do not depend on time t (33),

$$\lim_{T \rightarrow \infty} \langle f(x^i, t) \rangle_T = \langle f(x^i) \rangle_T = \langle f \rangle_E(x^i), \quad (92)$$

(2) its ensemble average $\langle f \rangle_E(x^i, t)$ (74) is equal to its space average $\langle f(x^i, t) \rangle_S$ (40) over the whole space $S = E^3$ and both averages do not depend on a position $\{x^i\}$ (46),

$$\lim_{V_S \rightarrow \infty} \langle f(x^i, t) \rangle_S = \langle f(t) \rangle_S = \langle f \rangle_E(t), \quad (93)$$

(3) its ensemble average $\langle f \rangle_E(x^i, t)$ (74) is equal to its space-time average $\langle f(x^i, t) \rangle_{ST}$ (9) over an infinitely large time interval T and over the whole space $S = E^3$ and both averages do not depend on on time t and a position $\{x^i\}$ (33) and (46),

$$\lim_{T \rightarrow \infty} \lim_{V_S \rightarrow \infty} \langle f(x^i, t) \rangle_{ST} = \langle f \rangle_{ST} = \langle f \rangle_E. \quad (94)$$

Thus, if proved, the Ergodic hypothesis of Newtonian cosmology (92), (93) and (94) guarantees that the values of cosmological fluid quantities measured by an observer in a Newtonian universe by means of space-time, time and/or space averagings are equal within the accuracy of measurements to the values predicted theoretically on the basis of the statistical treatment of the evolution of the Newtonian universe.

15 The Averaged Navier-Stokes-Poisson Equations in Kinematic Quantities

The system of the averaged Navier-Stokes-Poisson equations in kinematic quantities can be now established. Since the space-time (9) and ensemble averaging, (9) and (74), have the same analytic properties (19) and (81), satisfy the Reynolds conditions (17)-(18) and (82)-(84), and the same formulae for averaging out the material derivatives (64), (68) and (85), (87), the notation $\langle f \rangle(x^i, t)$ will be used hereafter for both averaging procedures. Both procedures lead to the same form of the averaged equations, though they are physically equivalent, strictly speaking, only under the Ergodic hypothesis of Newtonian cosmology (94), see Section 14. Use of the time and space averages (27), (40), see Sections 8 and 9, can be made every time when the physical conditions of a Newtonian universe require this and the corresponding parts of the Ergodic hypothesis of Newtonian cosmology, (92) and (93), are valid.

Theorem 6 (The averaged Navier-Stokes-Poisson equations) *If a covering (15) by the averaging time intervals T and the space regions S is determined through the cosmological fluid configuration, then the averaged system of the Navier-Stokes-Poisson equations in terms of the kinematic quantities (II-4), (II-6), (II-8), (II-12), (II-51)-(II-56), (II-59)-(II-61), (I-64)-(II-66), (II-69), (II-28) and (II-70) is as follows.*

The averaged Raychaudhuri evolution equation for the averaged expansion scalar $\langle\theta\rangle$

$$\frac{d\langle\theta\rangle}{dt} + \left\langle u^i \frac{\partial\theta}{\partial x^i} \right\rangle - \langle u^i \rangle \frac{\partial\langle\theta\rangle}{\partial x^i} + \frac{1}{3}\langle\theta^2\rangle + 2(\langle\sigma^2\rangle - \langle\omega^2\rangle) + 4\pi G\langle\rho\rangle - \Lambda - \delta^{ij}\langle A_i\rangle_{,j} = 0, \quad (95)$$

The averaged propagation equation for the averaged shear tensor $\langle\sigma_{ij}\rangle$

$$\begin{aligned} & \frac{d\langle\sigma_{ij}\rangle}{dt} + \left\langle u^k \frac{\partial\sigma_{ij}}{\partial x^k} \right\rangle - \langle u^k \rangle \frac{\partial\langle\sigma_{ij}\rangle}{\partial x^k} + \delta^{kl}\langle\sigma_{ik}\sigma_{lj}\rangle + \frac{2}{3}\langle\theta\sigma_{ij}\rangle - \\ & \frac{1}{3}\delta_{ij}(2\langle\sigma^2\rangle + \langle\omega^2\rangle - \delta^{kl}\langle A_k\rangle_{,l}) + \langle\omega_i\omega_j\rangle + \langle E_{ij}\rangle - \langle A_{(i}\rangle_{,j)} = 0, \end{aligned} \quad (96)$$

The averaged propagation equation for the averaged vorticity vector $\langle\omega^i\rangle$

$$\frac{d\langle\omega^i\rangle}{dt} + \left\langle u^j \frac{\partial\omega^i}{\partial x^j} \right\rangle - \langle u^j \rangle \frac{\partial\langle\omega^i\rangle}{\partial x^j} + \frac{2}{3}\langle\theta\omega^i\rangle - \delta^{ij}\langle\sigma_{jk}\omega^k\rangle - \frac{1}{2}\varepsilon^{ijk}\langle A_j\rangle_{,k} = 0, \quad (97)$$

The averaged kinematic decomposition of the averaged tensor $\langle u_{i,j}\rangle$

$$\langle u_i\rangle_{,j} = \langle\sigma_{ij}\rangle + \frac{1}{3}\delta_{ij}\langle\theta\rangle + \langle\omega_{ij}\rangle, \quad \langle\sigma_{ij}\rangle = \langle\sigma_{ji}\rangle, \quad \langle\theta\rangle = \delta^{ij}\langle\sigma_{ij}\rangle, \quad \langle\omega_{ij}\rangle = -\langle\omega_{ji}\rangle, \quad (98)$$

The averaged first identity for the averaged tensor $\langle u_{i,j}\rangle$

$$\langle u_i\rangle_{,jt} = \langle u_i\rangle_{,tj}, \quad (99)$$

The averaged second identity for the averaged tensor $\langle u_{i,j}\rangle$

$$\langle u_i\rangle_{,jk} = \langle u_i\rangle_{,kj}, \quad (100)$$

The averaged first constraint equation

$$\delta^{jk}(\langle \sigma_{ij} \rangle_{,k} - \langle \omega_{ij} \rangle_{,k}) - \frac{2}{3} \langle \theta \rangle_{,j} = 0, \quad (101)$$

The averaged second constraint equation

$$\langle \omega^i \rangle_{,i} = 0, \quad (102)$$

The averaged third constraint equation

$$\delta^{j(i}(\langle \sigma_{jk} \rangle_{,l} + \langle \omega_{jk} \rangle_{,l})\varepsilon^{m)kl} = 0, \quad (103)$$

The averaged first integrability condition

$$\langle E_i^j \rangle_{,j} = \frac{8\pi G}{3} \langle \rho \rangle_{,i}, \quad (104)$$

The averaged second integrability condition

$$\langle E_k^{(i} \rangle_{,j} \varepsilon^{l)kj} = 0, \quad (105)$$

The averaged third integrability condition

$$\langle \sigma_{[k}^{[i} \rangle_{,l]}^{,j]} + \frac{2}{3} \delta_{[k}^{[i} \langle \theta \rangle_{,l]}^{,j]} = 0, \quad (106)$$

The averaged equation of continuity as the evolution equation for the averaged fluid density $\langle \rho \rangle$

$$\frac{d \langle \rho \rangle}{dt} + \left\langle u^i \frac{\partial \rho}{\partial x^i} \right\rangle - \langle u^i \rangle \frac{\partial \langle \rho \rangle}{\partial x^i} + \langle \rho \theta \rangle = 0, \quad (107)$$

The averaged equation of state

$$\langle \rho \rangle = \langle \rho(p) \rangle, \quad (108)$$

The averaged total acceleration $\langle A_i \rangle$ by the averaged Navier-Stokes equation

$$\langle A_i \rangle = - \left\langle \frac{1}{\rho} p_{,i} \right\rangle, \quad (109)$$

The averaged Poisson equation for the averaged Newtonian gravitational potential $\langle \phi \rangle$

$$\delta^{ij} \langle g_i \rangle_{,j} = 4\pi G \langle \rho \rangle - \Lambda, \quad \text{or} \quad \delta^{ij} \langle \phi \rangle_{,ij} = 4\pi G \langle \rho \rangle - \Lambda, \quad (110)$$

The averaged identity for the averaged Newtonian gravitational acceleration $\langle g_i \rangle$

$$\varepsilon^{ijk} \langle g_j \rangle_{,k} = 0, \quad (111)$$

The averaged tidal force tensor $\langle E_{ij} \rangle$ for the averaged Newtonian gravitational potential $\langle \phi \rangle$

$$\langle E_{ij} \rangle = \langle \phi \rangle_{,ij} - \frac{1}{3} \delta_{ij} \delta^{kl} \langle \phi \rangle_{,kl}, \quad \langle E \rangle_{ij} = \langle E_{ji} \rangle, \quad \delta^{kl} \langle E_{kl} \rangle = 0, \quad (112)$$

The averaged evolution equations for the averaged tidal force tensor $\langle E_{ij} \rangle$

$$\begin{aligned} \frac{d \langle E_{ij} \rangle}{dt} + \left\langle u^k \frac{\partial E_{ij}}{\partial x^k} \right\rangle - \langle u^k \rangle \frac{\partial \langle E_{ij} \rangle}{\partial x^k} + \langle \theta E_{ij} \rangle - \delta_{ij} \delta^{kn} \delta^{lm} \langle \sigma_{kl} E_{nm} \rangle - \\ 3\delta^{kl} \langle \sigma_{k(i} E_{j)l} \rangle - \delta^{kl} \langle \omega_{k(i} E_{j)l} \rangle + 4\pi G \langle \rho \sigma_{ij} \rangle = 0, \end{aligned} \quad (113)$$

Proof. The averaging out of the system of the Navier-Stokes-Poisson equations in terms of kinematic quantities (II-4), (II-6), (II-8), (II-12), (II-51)-(II-56), (II-59)-(II-61), (I-64)-(II-66), (II-69), (II-28) and (II-70) is straightforward by applying the Reynold conditions (17)-(18) or (82)-(84), and the formulae for averaging out the material derivatives (64), (68) or (85), (87). **QED**

The system of the averaged Navier-Stokes-Poisson (95)-(113) contains the following independent set of the single-point second order moment functions:

The single-point second order moments in the averaged Navier-Stokes-Poisson equations:

The fluid velocity u^i with the spatial derivatives of the kinematic quantities σ_{ij} , θ and ω^i , the fluid density ρ and the tidal force tensor E_{ij}

$$\left\langle u^i \frac{\partial \theta}{\partial x^i} \right\rangle, \quad \left\langle u^k \frac{\partial \sigma_{ij}}{\partial x^k} \right\rangle, \quad \left\langle u^j \frac{\partial \omega^i}{\partial x^j} \right\rangle, \quad \left\langle u^i \frac{\partial \rho}{\partial x^i} \right\rangle, \quad \left\langle u^k \frac{\partial E_{ij}}{\partial x^k} \right\rangle, \quad (114)$$

The kinematic quantities σ_{ij} , θ and ω^i themselves

$$\langle \theta^2 \rangle, \quad \delta^{kl} \langle \sigma_{ik} \sigma_{lj} \rangle, \quad \langle \theta \sigma_{ij} \rangle, \quad \langle \omega_i \omega_j \rangle, \quad \langle \theta \omega^i \rangle, \quad \delta^{ij} \langle \sigma_{jk} \omega^k \rangle, \quad (115)$$

The fluid density ρ with the kinematic quantities σ_{ij} and θ and the pressure gradient $p_{,i}$

$$\langle \rho \theta \rangle, \quad \langle \rho \sigma_{ij} \rangle, \quad \left\langle \frac{1}{\rho} p_{,i} \right\rangle, \quad (116)$$

The tidal force tensor E_{ij} with the kinematic quantities σ_{ij} , θ and ω^i

$$\langle \theta E_{ij} \rangle, \quad \delta_{ij} \delta^{km} \delta^{lm} \langle \sigma_{kl} E_{mn} \rangle, \quad \delta^{kl} \langle \sigma_{k(i} E_{j)l} \rangle, \quad \delta^{kl} \langle \omega_{k(i} E_{j)l} \rangle. \quad (117)$$

The system of the averaged Navier-Stokes-Poisson (95)-(113) is not closed with respect to these single-point second order moment functions. The system should be supplemented by the equations for these moment functions (114)-(117). A derivation of a system of the equations for the second moments is known [16]-[18], [20], [26], [28], [32], [33] to produce the higher-order moments of fluid fields. This is the so-called closure problem. To make an analysis possible and also take into account properties of particular fluid configurations one must formulate some conditions for the fluid field moments of some order to terminate this infinite set of equations.

Acknowledgments

Roustam Zalaletdinov would like to thank Alan Coley for his hospitality in Dalhousie University and acknowledges partial support of this work by the Swiss National Science Foundation, Grant 7BYPJ065731.

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